PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Hassitt-Type Young Operator Expansions I. An Orthogonal Transformation between (a) the Young Operators of the Symmetric Group S$_{A+1}$ and (b) the Two-Sided Products of the Young Operators of S $_{A}$ with the Transposition P $_{A,\,A+1}$

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HASSITT-TYPE YOUNG OPERATOR EXPANSIONS

I. AN ORTHOGONAL TRANSFORMATION BETWEEN (a) THE YOUNG OPERATORS OF THE SYMMETRIC GROUP S_{A+1} AND (b) THE TWO-SIDED PRODUCTS OF THE YOUNG OPERATORS OF S_A WITH THE TRANSPOSITION $P_{A,A+1}$

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CONTENTS

An expansion, of the type considered by Hassitt, of the Young operators of S_{A+1} in terms of the two-sided products of the Young operators of S_A into the transposition $P_{A,A+1}$, is derived in complete generality and explicitly evaluated for the symmetry type of the general Wigner nuclear supermultiplet. The expansion allows of a recursive construction of the Young operators which is simpler than that considered, for example, by Thrall. A complete set of Young operators for S_3, S_4, S_5 and S_6 is explicitly constructed and tabulated. In an appendix explicit formulae are given for the most general case.

1. Definitions and notation

The Young operators $o^p_{(u);(v)}$ of the symmetric group S_A are defined in terms of the irreducible orthogonal matrix representations of the group by (Young 1900–35)

$$o^p_{(u);(v)} = (f^p/A!)\sum_P \langle (u)|P|(v)\rangle P, \tag{1·1}$$

where the summation is over all the $A!$ permutations P of S_A. Here

$$\langle (u)|P|(v)\rangle = \langle (v)|P^{-1}|(u)\rangle \tag{1·2}$$

denotes the matrix element of P in the irreducible orthogonal representation R^p characterized by the regular partition p of A

$$p = [p_1 p_2 \dots p_k], \quad p_1 \geqslant p_2 \geqslant \dots \geqslant p_k, \quad p_1 + p_2 + \dots + p_k = A, \tag{1·3}$$

to which corresponds a Young tableau with k rows. An alternative notation for the same partition

$$p = [(p^{(1)})^{k_1} (p^{(2)})^{k_2} \dots (p^{(n)})^{k_n}], \quad k_1 p^{(1)} + k_2 p^{(2)} + \dots + k_n p^{(n)} = A, \tag{1·4}$$

$$p^{(1)} > p^{(2)} > \dots > p^{(n)}, \quad k_1 + k_2 + \dots + k_n = k, \tag{1·5}$$

uses exponents k_r $(r = 1, 2, \dots, n)$ in the usual way to denote repetition of the partition numbers. n is the number of *different* row lengths in the tableau. The partition p need not appear explicitly in the notation for the matrix elements because it is implicit in the row and column labels (u) and (v). Each of these labels takes on f^p distinct values, f^p being the dimension (number of rows in the square matrices) of the representation R^p. As is well known, the labels (u) may be put into one-to-one correspondence with the f^p standard allowed arrangements of the A integers $1, 2, \dots, A$ in the Young tableau p and hence may be described by Yamanouchi symbols (Jahn 1951)

$$(u) = u_1 u_2 \dots u_{A-1} u_A, \tag{1·6}$$

with u_B $(B = 1, 2, \dots, A)$ equal to the number of the row of the tableau in which the number B occurs in the given standard arrangement (u).

Since the number A must occur at the end of a row and at the bottom of a column in the Young tableau p, it follows that there are just n possible values for the row number u_A, namely

$$u_A = k_1 + k_2 + \dots + k_r \quad (r = 1, 2, \dots, n), \tag{1·7}$$

i.e. there are n distinct rows of the Young tableau p for which the last square may be removed leaving a regular Young tableau with $A-1$ squares. We denote the corresponding irreducible orthogonal representation of S_{A-1} and its dimension respectively by

$$R^p_s, \quad f^p_s \quad (s = k_1 + k_2 + \dots + k_r;\ r = 1, 2, \dots, n), \tag{1·8}$$

a typical Yamanouchi symbol for this representation being

$$(u') = u_1 u_2 \dots u_{A-1}, \tag{1·9}$$

such that

$$(u')\, s = (u) \tag{1·10}$$

is a Yamanouchi symbol of R^p. From a well known property of the representations of the symmetric group it follows that (Weyl 1946)

$$\sum_s f^p_s = f^p. \tag{1·11}$$

Considering now the process of adding one square to the Young tableau p to obtain a regular Young tableau with $A+1$ squares containing p as a constituent, i.e. the process of adding the row number u_{A+1} to the Yamanouchi symbol (u) of S_A

$$(u)\, u_{A+1} = u_1 u_2 \dots u_A u_{A+1}, \tag{1·12}$$

we see that there are just $n+1$ possible values for the row number u_{A+1} namely

$$u_{A+1} = k_0 + k_1 + k_2 + \dots + k_r \quad (r = 0, 1, 2, \dots, n;\ k_0 = 1). \tag{1·13}$$

We denote the corresponding irreducible orthogonal representation of S_{A+1} and its dimension respectively by

$$R^{pt}, \quad f^{pt} \quad (t = k_0+k_1+k_2+\ldots+k_r;\ r = 0, 1, 2, \ldots, n;\ k_0 = 1). \tag{1·14}$$

Another well known property of the representations of the symmetric group then finds expression in the equation (Weyl 1946)

$$\sum_t f^{pt} = (A+1) f^p. \tag{1·15}$$

The non-vanishing matrix elements of $P_{A,A+1}$ in the representations R^{pt} of S_{A+1} are given by (Young 1900–35)

$$\langle (u')\, st|\, P_{A,A+1}\, |(u')\, st\rangle = r^{pt}_{st}, \tag{1·16}$$

$$\langle (u')\, st|\, P_{A,A+1}\, |(u')\, ts\rangle = \langle (u')\, ts|\, P_{A,A+1}\, |(u')\, st\rangle = \{1-(r^{pt}_{st})^2\}^{\frac{1}{2}} \quad (s \neq t), \tag{1·17}$$

where

$$1/r^{pt}_{st} = a^{pt}_{st} = p_t+1-p_s+s-t, \tag{1·18}$$

a^{pt}_{st} being the Young axial distance from the added last square in row t to the last square of row s in the Young tableau p. We see that r^{pt}_{st} is positive if $s > t$, negative if $s < t$ and that

$$r^{pt}_{tt} = 1. \tag{1·19}$$

2. The transformation and direct verification of its orthogonality

Our aim is to establish the following expansion of the Young operators of S_{A+1} in terms of the Young operators of S_A and the single transposition $P_{A,A+1}$:

$$\{(A+1) f^{pt}_s/f^{pt}\}^{\frac{1}{2}}\, o^{pt}_{(u)t;(w)s} = o^{p's}_{(u);(v')s}\, (A f^{p'}/f^p)^{\frac{1}{2}}\, P_{A,A+1}\, o^{p't}_{(v')t;(w)} \quad (t \neq s), \tag{2·1}$$

$$\{(A+1) f^{p}/f^{pt}\}^{\frac{1}{2}}\, o^{pt}_{(u)t;(v)t} = [f^{pt}/\{(A+1) f^p\}]^{\frac{1}{2}}\, o^p_{(u);(v)} + \sum_s r^{pt}_{st} \left\{\frac{A f^{pt} f^p_s}{(A+1)\,(f^p)^2}\right\}^{\frac{1}{2}} o^{p's}_{(u);(v')s} \left(\frac{A f^{p'}}{f^p}\right)^{\frac{1}{2}} P_{A,A+1}\, o^{p's}_{(v')s;(v)}, \tag{2·2}$$

and to show, moreover, that (2·2) has the form of an orthogonal transformation. In both (2·1) and (2·2) p' is a partition of $A-1$ and (v') is *any* Yamanouchi symbol of the corresponding representation $R^{p'}$ of S_{A-1} (the fact that (v') in the two Young operators on either side of $P_{A,A+1}$ is arbitrary is due to the fact that $P_{A,A+1}$ commutes with the permutations of S_{A-1}). In (2·1) (u) is a Yamanouchi symbol of the representation R^p of S_A and (w) is a Yamanouchi symbol of a different representation R^{pt}_s, $s \neq t$, of S_A. In (2·2) (u) and (v) are possibly different Yamanouchi symbols belonging to one and the same representation R^p of S_A. The numerical factor $\{Af^{p'}/f^p\}^{\frac{1}{2}}$ is inserted in front of $P_{A,A+1}$ in connexion with the orthogonality and has the effect of normalizing the two-sided product operators. In (2·2) since $R^{p's} = R^p$, p' varies with s in the summation; in (2·1) we also have $R^{p's} = R^p$ but $R^{p't} = R^{pt}_s \neq R^p$.

In order to exhibit (2·2) as an orthogonal transformation we define for given p, (u) and (v) the $n+1$ operators

$$0_t(p, (u), (v)) = \{(A+1) f^p/f^{pt}\}^{\frac{1}{2}}\, o^{pt}_{(u)t;(v)t} \quad (t = k_0+k_1+\ldots+k_r;\ r = 0, 1, \ldots, n;\ k_0 = 1) \tag{2·3}$$

and the n operators

$$T_s(p, (u), (v)) = o^p_{(u);(v')s}\, (A f^p_s/f^p)^{\frac{1}{2}}\, P_{A,A+1}\, o^p_{(v')s,(v)} \quad (s = k_1+k_2+\ldots+k_r;\ r = 1, 2, \ldots, n) \tag{2·4}$$

and may then present (2·2) in the form

$$0_t(p,(u),(v)) = c^p_{0t}o^p_{(u);(v)} + \sum_s c^p_{st}T_s(p,(u),(v)), \tag{2·5}$$

with an $(n+1)$-rowed square matrix of coefficients

$$c^p_{0t} = [f^{pt}/\{(A+1)f^p\}]^{\frac{1}{2}}, \tag{2·6}$$

$$c^p_{st} = r^{pt}_{st}[Af^{pt}f^p_s/\{(A+1)(f^p)^2\}]^{\frac{1}{2}}. \tag{2·7}$$

The normalization condition

$$(c^p_{0t})^2 + \sum_s (c^p_{st})^2 = 1 \tag{2·8}$$

requires

$$\sum_s (r^{pt}_{st})^2 f^p_s = f^p\{(A+1)f^p - f^{pt}\}/(Af^{pt}). \tag{2·9}$$

To establish (2·9) we show below that

$$(r^{pt}_{st})^2 = 1-(A+1)f^{pt}_s f^p/(Af^{pt}f^p_s) \quad (s \neq t), \tag{2·10}$$

where f^{pt}_s is zero when the corresponding partition of A is not an allowed one. This gives

$$\sum_s (r^{pt}_{st})^2 f^p_s = f^p_t + \sum_{s\neq t} f^p_s - \{(A+1)f^p/(Af^{pt})\}\sum_{s\neq t} f^{pt}_s \tag{2·11}$$

and (2·9) follows from the standard relations

$$\sum_s f^p_s = f^p, \quad \sum_s f^{pt}_s = f^{pt}. \tag{2·12}$$

To prove (2·10) we start from the dimension formula (Weyl 1946)

$$f^p = A!\,D^p/(m_1!\,m_2!\dots m_k!), \quad m_r = p_r+k-r \quad (r = 1,2,\dots,k), \tag{2·13}$$

where

$$D^p = D(m_1,m_2,\dots,m_k) = \prod_{\substack{r<s\\1}}^{k}(m_r-m_s). \tag{2·14}$$

From (2·13) we deduce, for the case $t \neq k+1$,

$$(A+1)f^{pt}_s f^p/(Af^{pt}f^p_s) = D^{pt}_s D^p/(D^{pt}D^p_s), \tag{2·15a}$$

where, taking the case $s<t$,

$$D^{pt}_s = D(m_1,m_2,\dots,m_s-1,\dots,m_t+1,\dots,m_k), \tag{2·16a}$$

$$D^p = D(m_1,m_2,\dots,m_s,\dots,m_t,\dots,m_k), \tag{2·17}$$

$$D^{pt} = D(m_1,m_2,\dots,m_s,\dots,m_t+1,\dots,m_k), \tag{2·18a}$$

$$D^p_s = D(m_1,m_2,\dots,m_s-1,\dots,m_t,\dots,m_k). \tag{2·19}$$

It follows

$$\begin{aligned}(A+1)f^{pt}_s f^p/(Af^{pt}f^p_s) &= (m_s-1-m_t-1)(m_s-m_t)/(m_s-m_t-1)(m_s-1-m_t)\\ &= \{(m_s-m_t-1)^2-1\}/(m_s-m_t-1)^2\\ &= \{(a^{pt}_{st})^2-1\}/(a^{pt}_{st})^2,\end{aligned} \tag{2·20a}$$

since the Young axial distance a^{pt}_{st} is given by

$$a^{pt}_{st} = p_t+1-p_s+s-t = m_t+1-m_s. \tag{2·21}$$

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Since only the square of the axial distance enters into (2·20) it is clear that this relation holds also in the case $s > t$. Thus

$$(A+1)\,f_s^{pt}f^p/(Af^{pt}f_s^p) = 1-1/(a_{st}^{pt})^2$$
$$= 1-(r_{st}^{pt})^2 \quad (s \neq t), \tag{2·22}$$

establishing (2·10) for the case $t \neq k+1$. For the case $t = k+1$ we find

$$(A+1)\,f_s^{p,k+1}f^p/(Af^{p,k+1}f_s^p) = (m_s+1)\,D_s^{p,k+1}D^p/(m_s D^{p,k+1}D_s^p) \tag{2·15b}$$
$$= (m_s+1)\,(m_s-1)/m_s^2, \tag{2·20b}$$

since

$$D_s^{p,k+1} = D(m_1+1, m_2+1, \ldots, m_s, \ldots, m_k+1, 1) = D(m_1, m_2, \ldots, m_s-1, \ldots, m_k, 0) \tag{2·16b}$$

and

$$D^{p,k+1} = D(m_1+1, m_2+1, \ldots, m_s+1, \ldots, m_k+1, 1) = D(m_1, m_2, \ldots, m_s, \ldots, m_k, 0). \tag{2·18b}$$

It follows, by (2·30) below, that (2·10) is universally valid. Since r_{st}^{pt} is positive or negative according as s is greater or less than t, we deduce further

$$r_{st}^{pt} = +[1-(A+1)f_s^{pt}f^p/(Af^{pt}f_s^p)]^{\frac{1}{2}} \quad (s > t), \tag{2·23}$$
$$r_{st}^{pt} = -[1-(A+1)f_s^{pt}f^p/(Af^{pt}f_s^p)]^{\frac{1}{2}} \quad (s < t), \tag{2·24}$$

where f_s^{pt} is zero if the corresponding partition of A is not allowed.

The normalization condition for the column of coefficients c_{0t}^p follows immediately from (1·15)

$$\sum_t (c_{0t}^p)^2 = \sum_t f^{pt}/\{(A+1)f^p\} = 1. \tag{2·25}$$

Using the expression (2·10) for $(r_{st}^{pt})^2$ and the relations

$$\sum_{t\neq s} f^{pt} = (A+1)f^p - f^{ps}, \quad \sum_{t\neq s} f_s^{pt} = Af_s^p - f^p, \tag{2·26}$$

we find also

$$\sum_t (c_{st}^p)^2 = Af^{ps}f_s^p/\{(A+1)\,(f^p)^2\} + Af_s^p\{(A+1)f^p - f^{ps}\}/\{(A+1)\,(f^p)^2\} - (Af_s^p - f^p)/f^p = 1, \tag{2·27}$$

establishing the normalization condition for the column of coefficients c_{st}^p also. The orthogonality of the first column of coefficients with the remaining columns requires

$$\sum_t c_{0t}^p c_{st}^p = 0 = \sum_t r_{st}^{pt} f^{pt}. \tag{2·28}$$

To establish this relation we write it in the form

$$0 = f^{p,k+1}/a_{s,k+1}^{p,k+1} + \sum_{t\neq k+1} f^{pt}/a_{st}^{pt}, \tag{2·29}$$

with
$$a_{st}^{pt} = m_t - m_s + 1, \quad a_{s,k+1}^{p,k+1} = 1 - p_s + s - (k+1) = -m_s. \tag{2·30}$$

Since then from (1·15) we have

$$f^{p,k+1} = (A+1)f^p - \sum_{t\neq k+1} f^{pt}, \tag{2·31}$$

it follows (2·28) is established if we show

$$(A+1)f^p = \sum_{t\neq k+1} (m_t+1)f^{pt}/(m_t+1-m_s) \tag{2·32}$$

is valid for all values of s. Using the dimension formula (2·13) it may be seen that this is equivalent to showing that

$$D(m_1, m_2, \ldots, m_k) = \sum_{t=1}^{k} D(m_1, m_2, \ldots, m_{t-1}, m_t+1, m_{t+1}, \ldots, m_k)/(m_t+1-m_s) \qquad (2\cdot33)$$

is valid for $s = 1, 2, \ldots, k$. The validity of this set of k linear relations between alternating functions follows, however, from Cauchy's lemma (Weyl 1946)

$$\det|1/(a_t-b_s)| = D(a_1, a_2, \ldots, a_k)\, D(b_1, b_2, \ldots, b_k) \Big/ \left\{\prod_{s=1}^{k}\prod_{t=1}^{k}(a_t-b_s)\right\} \quad (s, t = 1, 2, \ldots, k), \qquad (2\cdot34)$$

for it is well known (Scott & Mathews 1904), and easily verified, that it follows from this lemma that the solution of the k linear equations

$$\sum_{t=1}^{k} y_t/(a_t-b_s) = 1 \quad (s = 1, 2, \ldots, k), \qquad (2\cdot35)$$

is given by $$y_t = \prod_{r=1}^{k}(a_t-b_r) \Big/ \left\{\prod_{i=1}^{t-1}\prod_{j=t+1}^{k}(a_t-a_i)\,(a_t-a_j)\right\} \quad (t = 1, 2, \ldots, k). \qquad (2\cdot36)$$

Hence putting

$$a_t = m_t+1, \quad b_s = m_s, \quad y_t = D(m_1, m_2, \ldots, m_t+1, \ldots, m_k)/D(m_1, m_2, \ldots, m_k), \qquad (2\cdot37)$$

the relations (2·33) are equivalent to

$$D(m_1, m_2, \ldots, m_{t-1}, m_t+1, m_{t+1}, \ldots, m_k)/D(m_1, m_2, \ldots, m_k)$$
$$= \prod_{i=1}^{t-1}\prod_{j=t+1}^{k}(m_t+1-m_i)\,(m_t+1-m_j) \Big/ \left\{\prod_{i=1}^{t-1}\prod_{j=t+1}^{k}(m_t-m_i)\,(m_t-m_j)\right\}, \qquad (2\cdot38)$$

the truth of which follows at once from the definition of the alternating function. Thus (2·33) and hence (2·28) has been proved.

To complete the direct proof of the orthogonality of the transformation (2·5) we have to show also that $$\sum_t c^p_{s_1t}c^p_{s_2t} = 0 = \sum_t r^{pt}_{s_1t}r^{pt}_{s_2t}f^{pt} \quad \text{for} \quad s_1 \neq s_2. \qquad (2\cdot39)$$

Treating this in the same manner as (2·28) we find this requires

$$(A+1)f^p = \sum_{t=1}^{k} f^{pt}[1-m_{s_1}m_{s_2}/\{(m_t-m_{s_1}+1)\,(m_t-m_{s_2}+1)\}], \qquad (2\cdot40)$$

equivalent, through the dimension formula, to the algebraic identity

$$D(m_1, \ldots, m_k) = \sum_{t=1}^{k} D(m_1, \ldots, m_t+1, \ldots, m_k)\,(m_t+1-m_{s_1}-m_{s_2})/\{(m_t+1-m_{s_1})\,(m_t+1-m_{s_2})\} \qquad (2\cdot41)$$

between alternating functions, to be shown valid for $s_1 \neq s_2$ and $s_1, s_2 = 1, 2, \ldots, k$. In fact, the partial fraction equality, for $s_1 \neq s_2$

$$1/\{(m_t+1-m_{s_1})\,(m_t+1-m_{s_2})\} = \{1/(m_t-m_{s_1}+1) - 1/(m_t-m_{s_2}+1)\}/(m_{s_1}-m_{s_2}), \qquad (2\cdot42)$$

enables us to write (2·41) in the form

$$D(m_1, m_2, \ldots, m_k)$$
$$= \sum_{t=1}^{k}\left[\frac{1}{m_t+1-m_{s_2}} - \frac{m_{s_2}}{m_{s_1}-m_{s_2}}\left\{\frac{1}{m_t-m_{s_1}+1} - \frac{1}{m_t-m_{s_2}+1}\right\}\right] D(m_1, \ldots, m_t+1, \ldots, m_k), \qquad (2\cdot43)$$

and the truth of this is apparent from (2·33). Hence (2·39) has been established and combining this with the normalization relation we have thus shown that

$$\sum_t r^{pt}_{s_1 t} r^{pt}_{s_2 t} f^{pt} = \delta(s_1, s_2)\,(A+1)\,(f^p)^2/(Af^p_{s_1}). \tag{2·44}$$

The above proof of the orthonormality of the columns of coefficients in (2·5) establishes at the same time the orthonormality of the rows of coefficients, so that we have also shown that

$$c^p_{0t_1} c^p_{0t_2} + \sum_s c^p_{st_1} c^p_{st_2} = \delta(t_1, t_2), \tag{2·45}$$

which is equivalent to the relation

$$\sum_s r^{pt_1}_{st_1} r^{pt_2}_{st_2} f^p_s = -f^p/A + \delta(t_1, t_2)\,(A+1)\,(f^p)^2/(Af^{pt_1}). \tag{2·46}$$

3. The inverse transformation

The orthogonality, which has just been established, of the matrix of coefficients in (2·5), enables us to present the inverse relation in the form

$$o^p_{(u);(v)} = \sum_t c^p_{0t}\, 0_t(p, (u), (v)), \tag{3·1}$$

$$T_s(p, (u), (v)) = \sum_t c^p_{st}\, 0_t(p, (u), (v)), \tag{3·2}$$

from which follow

$$o^p_{(u);(v)} = \sum_t o^{pt}_{(u)t;(v)t}, \tag{3·3}$$

$$o^p_{(u);(v')s} P_{A,A+1}\, o^p_{(v')s;(v)} = \sum_t r^{pt}_{st}\, o^{pt}_{(u)t;(v)t}, \tag{3·4}$$

to which we may add, from (2·1),

$$\begin{aligned} o^{p's}_{(u);(v')s} P_{A,A+1}\, o^{p't}_{(v')t;(w)} &= \{(A+1) f^{pt}_s f^p/(Af^{pt} f^p_s)\}^{\frac{1}{2}}\, o^{pt}_{(u)t;(w)s} \\ &= \{1-(r^{pt}_{st})^2\}^{\frac{1}{2}}\, o^{pt}_{(u)t;(w)s} \quad (s \neq t), \end{aligned} \tag{3·5}$$

where we have made use of (2·10). When we insert the matrix elements of $P_{A,A+1}$ from (1·16), (1·17) these take the form

$$o^{p's}_{(u);(v')s} P_{A,A+1}\, o^{p't}_{(v')t;(w)} = \langle (v')\,st|\, P_{A,A+1}\,|(v')\,ts\rangle\, o^{p'st}_{(u)t;(w)s} \quad (s \neq t), \tag{3·6}$$

$$o^{p's}_{(u);(v')s} P_{A,A+1}\, o^{p's}_{(v')s;(v)} = \sum_t \langle (v')\,st|\, P_{A,A+1}\,|(v')\,st\rangle\, o^{p'st}_{(u)t;(v)t}, \tag{3·7}$$

to which we add again

$$o^p_{(u);(v)} = \sum_t o^{pt}_{(u)t;(v)t}. \tag{3·8}$$

The orthogonality proof of § 2 shows that a proof of (3·6), (3·7) and (3·8) is a proof also of equations (2·1) and (2·2). We proceed now to a direct proof of these inverse equations.

The relation reciprocal to (1·1) is

$$P = \sum_{p(u)(v)} \langle (u)|\, P\,|(v)\rangle\, o^p_{(u);(v)}, \tag{3·9}$$

where P is any permutation of S_A and the summation is a double sum over all the Yamanouchi symbols of each representation R^p of S_A. Taking into account the complete set of non-vanishing matrix elements of $P_{A,A+1}$ given in (1·16) and (1·17) we deduce that

$$P_{A,A+1} = \sum_{p'(v')st} \langle (v')\,st|\, P_{A,A+1}\,|(v')\,st\rangle\, o^{p'st}_{(v')st;(v')st} + \sum_{\substack{p'(v')st \\ s \neq t}} \langle (v')\,st|\, P_{A,A+1}\,|(v')\,ts\rangle\, o^{p'st}_{(v')st;(v')ts}, \tag{3·10}$$

where the summation is over all allowed partitions p' of S_{A-1}, all allowed Yamanouchi symbols (v') of $R^{p'}$ and over all row numbers s and t for which addition of a square to row s and to row t of the Young tableau of $R^{p'}$ is permitted, addition of two squares to the same row being excluded from the second sum but not from the first. Applying (3·9) to the identity I we deduce also

$$I=\sum_{p'(v')st} o^{p'st}_{(v')st;\,(v')st}=\sum_{p(v)t} o^{pt}_{(v)t;\,(v)t}. \tag{3·11}$$

Inserting (3·10) into (3·6) we have then, with $s \neq t$,

$$\begin{aligned} o^{p's}_{(u);\,(v')s}P_{A,\,A+1}\,o^{p't}_{(v')t;\,(w)} = &\sum_{p'_1(v'_1)s_1t_1}\langle (v'_1)\,s_1t_1|\,P_{A,\,A+1}\,|(v'_1)\,s_1t_1\rangle\, o^{p's}_{(u);\,(v')s}\,o^{p'_1s_1t_1}_{(v'_1)s_1t_1;\,(v'_1)s_1t_1}\,o^{p't}_{(v')t;\,(w)} \\ &+\sum_{\substack{p'_1(v'_1)s_1t_1\\ s_1\neq t_1}}\langle (v'_1)\,s_1t_1|\,P_{A,\,A+1}\,|(v'_1)\,t_1s_1\rangle\, o^{p's}_{(u);\,(v')s}\,o^{p'_1s_1t_1}_{(v'_1)s_1t_1;\,(v'_1)t_1s_1}\,o^{p't}_{(v')t;\,(w)}. \end{aligned} \tag{3·12}$$

Now the Young operators of S_A have the properties (see Note at end of this paragraph)

$$o^{p}_{(u);\,(v)}\,o^{q}_{(r);\,(s)}=\delta(p,q)\,\delta((v),(r))\,o^{p}_{(u);\,(s)}, \tag{3·13}$$

$$o^{pt}_{(u)s;\,(v)t}=o^{pt/s}_{(u);\,(x)}\,o^{pt}_{(x)s;\,(v)t}=o^{pt}_{(u)s;\,(y)t}\,o^{p}_{(y);\,(v)}, \tag{3·14}$$

where (x) is any Yamanouchi symbol of R^{pt}_s and (y) any Yamanouchi symbol of R^p. It follows that

$$o^{p's}_{(u);\,(v')s}\,o^{p'_1s_1t_1}_{(v'_1)s_1t_1;\,(v'_1)s_1t_1}\,o^{p't}_{(v')t;\,(w)}, \tag{3·15}$$

involving the factors $\delta(s_1,s)$, $\delta(s_1,t)$ is zero for $s\neq t$, whilst

$$o^{p's}_{(u);\,(v')s}\,o^{p'_1s_1t_1}_{(v'_1)s_1t_1;\,(v'_1)t_1s_1}\,o^{p't}_{(v')t;\,(w)}=\delta(p'_1,p')\,\delta((v'_1),(v'))\,\delta(s_1,s)\,\delta(t_1,t)\,o^{p'st}_{(u)t;\,(w)s}. \tag{3·16}$$

It follows that for $s \neq t$

$$o^{p's}_{(u);\,(v')s}P_{A,\,A+1}\,o^{p't}_{(v')t;\,(w)}=\langle (v')\,st|\,P_{A,\,A+1}\,|(v')\,ts\rangle\,o^{p'st}_{(u)t;\,(w)s}, \tag{3·17}$$

establishing (3·6).

Inserting (3·10) into (3·7), on the other hand, we find

$$\begin{aligned} o^{p's}_{(u);\,(v')s}P_{A,\,A+1}\,o^{p's}_{(v')s;\,(v)} = &\sum_{p'_1(v'_1)s_1t_1}\langle (v'_1)\,s_1t_1|\,P_{A,\,A+1}\,|(v'_1)\,s_1t_1\rangle\, o^{p's}_{(u);\,(v')s}\,o^{p'_1s_1t_1}_{(v'_1)s_1t_1;\,(v'_1)s_1t_1}\,o^{p's}_{(v')s;\,(v)} \\ &+\sum_{\substack{p'_1(v'_1)s_1t_1\\ s_1\neq t_1}}\langle (v'_1)\,s_1t_1|\,P_{A,\,A+1}\,|(v'_1)\,t_1s_1\rangle\, o^{p's}_{(u);\,(v')s}\,o^{p'_1s_1t_1}_{(v'_1)s_1t_1;\,(v'_1)t_1s_1}\,o^{p's}_{(v')s;\,(v)} \end{aligned} \tag{3·18}$$

and we see that the last sum now vanishes since it is restricted to $s_1 \neq t_1$ whilst the product of the three Young operators in each term of the sum involves the factors $\delta(s_1,s)$, $\delta(t_1,s)$. The product of the three Young operators in the first sum is

$$o^{p's}_{(u);\,(v')s}\,o^{p'_1s_1t_1}_{(v'_1)s_1t_1;\,(v'_1)s_1t_1}\,o^{p's}_{(v')s;\,(v)}=\delta(p'_1,p')\,\delta((v'_1),(v'))\,\delta(s_1,s)\,o^{p'st_1}_{(u)t_1;\,(v)t_1}. \tag{3·19}$$

It follows

$$o^{p's}_{(u);\,(v')s}P_{A,\,A+1}\,o^{p's}_{(v')s;\,(v)}=\sum_t\langle (v')\,st|\,P_{A,\,A+1}\,|(v')\,st\rangle\,o^{p'st}_{(u)t;\,(v)t}, \tag{3·20}$$

which establishes (3·7).

Finally, multiplying both sides of (3·11) by $o^{p}_{(u);\,(v)}$ we have

$$o^{p}_{(u);\,(v)}=\sum_{p_1(v_1)t_1} o^{p}_{(u);\,(v)}\,o^{p_1t_1}_{(v_1)t_1;\,(v_1)t_1} \tag{3·21}$$

and the product of the Young operators on the right-hand side is

$$o^{p}_{(u);\,(v)}\,o^{p_1t_1}_{(v_1)t_1;\,(v_1)t_1}=\delta(p_1,p)\,\delta((v_1),(v))\,o^{pt_1}_{(u)t_1;\,(v)t_1}; \tag{3·22}$$

consequently

$$o^{p}_{(u);\,(v)}=\sum_t o^{pt}_{(u)t;\,(v)t}, \tag{3·23}$$

establishing (3·8).

With this direct proof of the inverse equations we have established also, through the proofs of § 2, the transformations (2·1), (2·2) between the Young operators of S_{A+1} and the two-sided products of the Young operators of S_A with $P_{A,\,A+1}$. A direct proof of the transformations (2·1), (2·2) is given in the next section.

Note

Relation (3·13) is a standard property of Young operators. (3·14) is established as follows. In

$$o^{pt}_{(x)s\,;\,(v)t} = \{f^{pt}/(A+1)\,!\} \sum_P \langle (x)\, s |\, P \,|\, (v)\, t\rangle\, P, \tag{3·24}$$

we write

$$P = RQ, \tag{3·25}$$

where R runs over the elements of S_A and

$$Q = I, \quad P_{j,\,A+1} \quad (j = 1, 2, \ldots, A). \tag{3·26}$$

Then

$$\begin{aligned} o^{pt}_{(x)s\,;\,(v)t} &= \{f^{pt}/(A+1)\,!\} \sum_{RQ(z)\,r} \langle (x)\, s |\, R \,|\, (z)\, r\rangle \langle (z)\, r |\, Q \,|\, (v)\, t\rangle\, RQ \\ &= \{f^{pt}/(A+1)\,!\} \sum_{Q(z)} \{\sum_R \langle (x) |\, R \,|\, (z)\rangle\, R\} \langle (z)\, s |\, Q \,|\, (v)\, t\rangle\, Q \\ &= [f^{pt}/\{(A+1)\, f^{pt}_s\}] \sum_{Q(z)} o^{pt/s}_{(x)\,;\,(z)} \langle (z)\, s |\, Q \,|\, (v)\, t\rangle\, Q. \end{aligned} \tag{3·27}$$

It follows

$$\begin{aligned} o^{pt/s}_{(u)\,;\,(x)}\, o^{pt}_{(x)s\,;\,(v)t} &= [f^{pt}/\{(A+1)\, f^{pt}_s\}] \sum_{Q(z)} o^{pt/s}_{(u)\,;\,(z)} \langle (z)\, s |\, Q \,|\, (v)\, t\rangle\, Q \\ &= o^{pt}_{(u)s\,;\,(v)t}. \end{aligned} \tag{3·28}$$

The relation

$$o^{pt}_{(u)s\,;\,(y)t}\, o^{p}_{(y)\,;\,(v)} = o^{pt}_{(u)s\,;\,(v)t} \tag{3·29}$$

follows in a similar manner from

$$o^{pt}_{(u)s\,;\,(y)t} = [f^{pt}/\{f^{p}(A+1)\}] \sum_{Q(w)} \langle (u)\, s |\, Q \,|\, (w)\, t\rangle\, Q\, o^{p}_{(w)\,;\,(y)}, \tag{3·30}$$

which is established in the same way.

4. Direct derivation of the transformation

The method of deriving relations (2·1), (2·2) given in § 3 is indirect since it involves the orthogonality verification of § 2. A direct proof (which was the one used in the original derivation) is as follows. Making use of the relations

$$\{(A+1)\, f^{pt}_s f^{p}/(A f^{pt} f^{p}_s)\}^{\frac{1}{2}} = \{1-(r^{pt}_{st})^2\}^{\frac{1}{2}} = \langle (v')\, st |\, P_{A,\,A+1} \,|\, (v')\, ts\rangle \quad (s \neq t), \tag{4·1}$$

$$r^{pt}_{st} = \langle (v')\, st |\, P_{A,\,A+1} \,|\, (v')\, st\rangle, \tag{4·2}$$

we may write equations (2·1), (2·2) in the form

$$\{(A+1)\, f^{pt}_s/f^{pt}\}\, o^{pt}_{(u)t\,;\,(w)s} = (A f^{p}_s/f^{p}) \langle (v')\, st |\, P_{A,\,A+1} \,|\, (v')\, ts\rangle\, o^{p's}_{(u)\,;\,(v')s} P_{A,\,A+1}\, o^{p't}_{(v')t\,;\,(w)} \quad (s \neq t), \tag{4·3}$$

$$\{(A+1)\, f^{p}/f^{pt}\}\, o^{pt}_{(u)t\,;\,(v)t} = o^{p}_{(u)\,;\,(v)} + \sum_s (A f^{p}_s/f^{p}) \langle (v')\, st |\, P_{A,\,A+1} \,|\, (v')\, st\rangle\, o^{p's}_{(u)\,;\,(v')s} P_{A,\,A+1}\, o^{p's}_{(v')s\,;\,(v)}. \tag{4·4}$$

We see that the two relations can be collected into the single equation

$$\begin{aligned} \{(A+1)\, f^{pt}_s/f^{pt}\}\, o^{pt}_{(u)t\,;\,(v)s} = \delta(s,t)\, o^{p}_{(u)\,;\,(v)} + \sum_{xy} (A f^{p}_x/f^{p}) \langle (v')\, xt |\, P_{A,\,A+1} \,|\, (v')\, ys\rangle \\ \times\, o^{p'x}_{(u)\,;\,(v')x} P_{A,\,A+1}\, o^{p'y}_{(v')y\,;\,(v)}. \end{aligned} \tag{4·5}$$

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Here if $s \neq t$ we must have $x = s$, $y = t$ and if $s = t$ we must have $x = y$ (and this variable is denoted by s in (4·4)).

Starting from the definition

$$o^{pt}_{(u)t;(v)s} = \{f^{pt}/(A+1)!\} \sum_R \langle (u)\, t|\, R\, |(v)\, s\rangle\, R, \tag{4·6}$$

where R runs over all the $(A+1)!$ permutations of S_{A+1}, we introduce a coset subdivision of S_{A+1} with respect to the subgroup S_A, writing

$$o^{pt}_{(u)t;(v)s} = \{f^{pt}/(A+1)!\} \sum_{j=1}^{A+1} \sum_P \langle (u)\, t|\, Q_j P\, |(v)\, s\rangle\, Q_j P, \tag{4·7}$$

where $$Q_j = R_{j,\,A+1} \quad (j = 1, 2, \ldots, A), \quad Q_{A+1} = I, \tag{4·8}$$

and P runs over all the permutations of S_A. We have then

$$o^{pt}_{(u)t;(v)s} = \{f^{pt}/(A+1)!\} \sum_{j=1}^{A+1} \sum_P \sum_{(w)r} \langle (u)\, t|\, Q_j\, |(w)\, r\rangle \langle (w)\, r|\, P\, |(v)\, s\rangle\, Q_j P. \tag{4·9}$$

Since, however, $$\langle (w)\, r|\, P\, |(v)\, s\rangle = \delta(r,s)\, \langle (w)\, |\, P\, |(v)\rangle \tag{4·10}$$

and $$o^q_{(w);(v)} = (f^q/A!) \sum_P \langle (w)\, |\, P\, |(v)\rangle\, P, \tag{4·11}$$

where $$R^q = R^{pt}_s, \quad f^q = f^{pt}_s, \tag{4·12}$$

we deduce $$o^{pt}_{(u)t;(v)s} = [f^{pt}/\{(A+1)f^{pt}_s\}] \sum_{j=1}^{A+1} \sum_{(w)} \langle (u)\, t|Q_j\, |(w)\, s\rangle\, Q_j\, o^q_{(w);(v)}. \tag{4·13}$$

Since now $$Q_j = R_{j,\,A+1} = P_{jA} P_{A,\,A+1} P_{jA} \quad (j = 1, 2, \ldots, A), \quad P_{AA} = I \tag{4·14}$$

and

$$\langle (u)\, t|\, P_{jA} P_{A,\,A+1} P_{j,\,A}\, |(w)\, s\rangle = \sum_{(x)} \sum_{(y)} \langle (u)\, |\, P_{jA}\, |(x)\rangle \langle (x)\, t|\, P_{A,\,A+1}\, |(y)\, s\rangle \langle (y)\, |\, P_{jA}\, |(w)\rangle$$
$$(j = 1, 2, \ldots, A), \quad P_{AA} = I, \tag{4·15}$$

we deduce

$$\{(A+1) f^{pt}_s / f^{pt}\}\, o^{pt}_{(u)t;(v)s} = \delta(s,t)\, o^p_{(u);(v)}$$
$$+ \sum_{j=1}^{A} \sum_{(w)(x)(y)} \langle (u)\, |\, P_{jA}\, |(x)\rangle \langle (x)\, t|\, P_{A,\,A+1}\, |(y)\, s\rangle \langle (y)\, |\, P_{jA}\, |(w)\rangle P_{jA} P_{A,\,A+1} P_{jA}\, o^q_{(w);(v)}. \tag{4·16}$$

Now from the standard property

$$P o^p_{(u);(v)} = \sum_{(x)} o^p_{(x);(v)} \langle (x)\, |\, P\, |(u)\rangle \tag{4·17}$$

of the Young operators of S_A, we deduce

$$P_{jA}\, o^q_{(w);(v)} = \sum_{(z)} o^q_{(z);(v)} \langle (z)\, |\, P_{jA}\, |(w)\rangle \quad (j = 1, 2, \ldots, A), \tag{4·18}$$

and using the orthogonality relation

$$\sum_{(w)} \langle (y)\, |\, P_{jA}\, |(w)\rangle \langle (z)\, |\, P_{jA}\, |(w)\rangle = \delta((y),(z)) \tag{4·19}$$

of the representation matrices for the transposition P_{jA}, we find

$$\{(A+1) f^{pt}_s / f^{pt}\}\, o^{pt}_{(u)t;(v)s} = \delta(s,t)\, o^p_{(u);(v)}$$
$$+ \sum_{(x)(y)} \langle (x)\, t|\, P_{A,\,A+1}\, |(y)\, s\rangle \left\{ \sum_{j=1}^{A} \langle (u)\, |\, P_{jA}\, |(x)\rangle\, P_{jA} \right\} P_{A,\,A+1}\, o^q_{(y);(v)}. \tag{4·20}$$

Writing now
$$(x) = (x')\,x, \quad (y) = (x')\,y, \tag{4·21}$$
so that (x') is a Yamanouchi symbol of the representation
$$R^{p'} = R^{p}_{x} = R^{q}_{y} = R^{pt}_{sy} \tag{4·22}$$
and using the fact that
$$\langle (x')\,xt|\, P_{A,\,A+1}\,|(x')\,ys\rangle = \langle (v')\,xt|\, P_{A,\,A+1}\,|(v')\,ys\rangle, \tag{4·23}$$
where (v') is any Yamanouchi symbol of $R^{p'}$, we deduce
$$\{(A+1)\,f^{pt}_{s}/f^{pt}\}\,o^{pt}_{(u)t;\,(v)s} = \delta(s,t)\,o^{p}_{(u);\,(v)} + \sum_{xy} \langle (v')\,xt|\, P_{A,\,A+1}\,|(v')\,ys\rangle \sum_{(x')} \sum_{j=1}^{A} \langle (u)\,|\, P_{jA}\,|(x')\,x\rangle\, P_{jA}\,P_{A,\,A+1}\,o^{q}_{(x')y;\,(v)}, \tag{4·24}$$
where the matrix element of $P_{A,\,A+1}$ has been taken out of the sum over (x'). Writing then
$$o^{q}_{(x')y;\,(v)} = o^{p'}_{(x');\,(v')}\,o^{p'y}_{(v')y;\,(v)}, \tag{4·25}$$
we have, since $o^{p'}_{(x');\,(v')}$ commutes with $P_{A,\,A+1}$,
$$\{(A+1)\,f^{pt}_{s}/f^{pt}\}\,o^{pt}_{(u)t;\,(v)s} = \delta(s,t)\,o^{p}_{(u);\,(v)} + \sum_{xy} \langle (v')\,xt|\, P_{A,\,A+1}\,|(v')\,ys\rangle \times \Big[\sum_{(x')} \sum_{j=1}^{A} \langle (u)\,|\, P_{jA}\,|(x')\,x\rangle\, P_{jA}\,o^{p'}_{(x');\,(v')}\Big]\, P_{A,\,A+1}\,o^{p'y}_{(v')y;\,(v)}. \tag{4·26}$$
By applying (4·13) to the Young operators of S_A we see, however, that
$$\sum_{j=1}^{A} \sum_{(x')} \langle (u)\,|\, P_{jA}\,|(x')\,x\rangle\, P_{jA}\,o^{p'}_{(x');\,(v')} = (Af^{p'}/f^{p'x})\,o^{p'x}_{(u);\,(v')x}. \tag{4·27}$$
It follows that
$$\{(A+1)\,f^{pt}_{s}/f^{pt}\}\,o^{pt}_{(u)t;\,(v)s} = \delta(s,t)\,o^{p}_{(u);\,(v)} + \sum_{xy} (Af^{p'}/f^{p'x})\,\langle (v')\,xt|\, P_{A,\,A+1}\,|(v')\,ys\rangle\, o^{p'x}_{(u);\,(v')x}\,P_{A,\,A+1}\,o^{p'y}_{(v')y;\,(v)}, \tag{4·28}$$
which agrees with (4·5) since, from (4·22), we have
$$f^{p'} = f^{p}_{x}, \quad f^{p'x} = f^{p}. \tag{4·29}$$
Thus formulae (2·1) and (2·2) have been given a direct derivation.

5. Connexion with normalized coset coefficients

If we define 'normalized' Young operators $[o^{pt}_{(u)t;\,(v)s}]$ (Jahn 1954) for S_{A+1} by
$$[o^{pt}_{(u)t;\,(v)s}] = \{(A+1)!/f^{pt}\}^{\frac{1}{2}}\,o^{pt}_{(u)t;\,(v)s}, \tag{5·1}$$
then the two reciprocal relations
$$o^{pt}_{(u)t;\,(v)s} = \{f^{pt}/(A+1)!\} \sum_{R} \langle (u)\,t|\, R\,|(v)\,s\rangle\, R, \tag{5·2}$$
$$R = \sum_{p(u)t\,(v)s} \langle (u)\,t|\, R\,|(v)\,s\rangle\, o^{pt}_{(u)t;\,(v)s}, \tag{5·3}$$
between the $(A+1)!$ permutations R of S_{A+1} and the $\sum_{pt} (f^{pt})^2 = (A+1)!$ Young operators $o^{pt}_{(u)t;\,(v)s}$ of S_{A+1} take the form
$$[o^{pt}_{(u)t;\,(v)s}] = \sum_{R} [(u)\,t|\, R\,|(v)\,s]\, R, \tag{5·4}$$
$$R = \sum_{p(u)t\,(v)s} [(u)\,t|\, R\,|(v)\,s]\,[o^{pt}_{(u)t;\,(v)s}], \tag{5·5}$$

with the same 'normalized' representation coefficients

$$[(u)\,t|\,R\,|(v)\,s] = \{f^{pt}/(A+1)!\}^{\frac{1}{2}}\langle(u)\,t|\,R\,|(v)\,s\rangle, \tag{5·6}$$

occurring in the two reciprocal relations. It follows that these normalized representation coefficients, as is also well known from general representation theory, define an $(A+1)!$-rowed square orthogonal matrix satisfying the orthonormality relations

$$\sum_{p(u)t(v)s}[(u)\,t|\,R\,|(v)\,s]\,[(u)\,t|\,S\,|(v)\,s] = \delta(R,S)$$
$$= \{1/(A+1)!\}\sum_{p(u)t(v)s} f^{pt}\langle(u)\,t|\,R\,|(v)\,s\rangle\langle(u)\,t|\,S\,|(v)\,s\rangle, \tag{5·7}$$

$$\sum_R [(u)\,t|\,R\,|(v)\,s]\,[(w)\,z|\,R\,|(x)\,y] = \delta((u),(w))\,\delta(t,z)\,\delta((v),(x))\,\delta(s,y)$$
$$= \{f^{pt}/(A+1)!\}\sum_R\langle(u)\,t|\,R\,|(v)\,s\rangle\langle(w)\,z|\,R\,|(x)\,y\rangle. \tag{5·8}$$

If we introduce the coset decomposition of S_{A+1} with respect to the subgroup S_A, writing

$$R = T_j P \quad (j = 1, 2, \ldots, A+1), \quad T_{A+1} = I, \tag{5·9}$$

R being an element of S_{A+1}, P an element of S_A, T_j a selected element in the jth coset (the $(j+1)$th coset is the subgroup S_A itself), we deduce from (5·8) (with s and t interchanged),

$$\delta((u_1),(u))\,\delta((v_1),(v))\,\delta(s_1,s)\,\delta(t_1,t)$$
$$= \{f^{pt}/(A+1)!\}\sum_{j=1}^{A+1}\sum_P\langle(u)\,s|\,T_jP\,|(v)\,t\rangle\langle(u_1)\,s_1|\,T_jP\,|(v_1)\,t_1\rangle$$
$$= \{f^{pt}/(A+1)!\}\sum_{j=1}^{A+1}\sum_{(w)(w_1)}\langle(u)\,s|\,T_j\,|(w)\,t\rangle\langle(u_1)\,s_1|\,T_j\,|(w_1)\,t_1\rangle$$
$$\times\sum_P\langle(w)\,|\,P\,|(v)\rangle\langle(w_1)\,|\,P\,|(v_1)\rangle, \tag{5·10}$$

where (v) and (w) are Yamanouchi symbols of the representation R^p of S_A. Since, however, from (5·8) applied to S_A, we have

$$\sum_P\langle(w)\,|\,P\,|(v)\rangle\langle(w_1)\,|\,P\,|(v_1)\rangle = (A!/f^p)\,\delta((w_1),(w))\,\delta((v_1),(v)), \tag{5·11}$$

we deduce

$$\sum_{(w)}\sum_{j=1}^{A+1}[f^{pt}/\{(A+1)f^p\}]\,\langle(u)\,s|\,T_j\,|(w)\,t\rangle\langle(u_1)\,s_1|\,T_j\,|(w)\,t_1\rangle = \delta((u_1),(u))\,\delta(s_1,s)\,\delta(t_1,t). \tag{5·12}$$

Defining 'normalized' coset coefficients by

$$[(u)\,s|\,T_j\,|(w)\,t] = [f^{pt}/\{(A+1)f^p\}]^{\frac{1}{2}}\langle(u)\,s|\,T_j\,|(w)\,t\rangle$$
$$= C^p_{t(u)s;\,j(w)}, \tag{5·13}$$

we may express (5·12) in the form

$$\sum_{j=1}^{A+1}\sum_{(w)} C^p_{t(u)s;\,j(w)}\,C^p_{t_1(u_1)s_1;\,j(w)} = \delta(t_1,t)\,\delta((u_1),(u))\,\delta(s_1,s), \tag{5·14}$$

where (w) is a Yamanouchi symbol of R^p, (u) a Yamanouchi symbol of R^{pt}_s. Since, for given p, the indices t, (u), s take on $\sum_t f^{pt}$ distinct values, whilst the indices j, (w) take on $(A+1)f^p$ distinct values and since, from (1·15),

$$\sum_p f^{pt} = (A+1)f^p, \tag{5·15}$$

it follows, from (5·14), that the coefficients $C^p_{t(u)s;j(w)}$ form a square orthogonal matrix with $(A+1)f^p$ rows. It is easily seen that the orthogonality relation for the columns

$$\sum_{\substack{t(u)s \\ (p \text{ fixed})}} C^p_{t(u)s;j(w)} C^p_{t(u)s;k(v)} = \sum_{\substack{t(u)s \\ (p \text{ fixed})}} [(u)\,s|\,T_j\,|(w)\,t]\,[(u)\,s|\,T_k\,|(v)\,t]$$

$$= \sum_{\substack{t(u)s \\ (p \text{ fixed})}} [f^{pt}/\{(A+1)f^p\}]\,\langle(u)\,s|\,T_j\,|(w)\,t\rangle\langle(u)\,s|\,T_k\,|(v)\,t\rangle = \delta(j,k)\,\delta((w),(v)), \quad (5·16)$$

is consistent with the relation

$$\sum_{p(u)s\,(w)t} f^{pt}\langle(u)\,s|\,T_j\,|(w)\,t\rangle\langle(u)\,s|\,T_k\,|(w)\,t\rangle = (A+1)!\,\delta(j,k), \quad (5·17)$$

which follows from (5·7). For (5·16) with $(v) = (w)$, summed over the f^p values of (w), gives

$$\sum_{\substack{(u)s\,(w)t \\ (p \text{ fixed})}} f^{pt}\langle(u)\,s|\,T_j\,|(w)\,t\rangle\langle(u)\,s|\,T_k\,|(w)\,t\rangle = (A+1)\,(f^p)^2\,\delta(j,k) \quad (5·18)$$

and (5·17) follows since
$$\sum_p (f^p)^2 = A!. \quad (5·19)$$

The coset operator expansion (4·13), written in the form

$$o^{pt}_{(u)s;(v)t} = [f^{pt}/\{(A+1)f^p\}] \sum_{j=1}^{A+1} \sum_{(w)} \langle(u)\,s|\,T_j\,|(w)\,t\rangle\,T_j\,o^p_{(w);(v)}, \quad (5·20)$$

may hence be expressed as

$$\{(A+1)f^p/f^{pt}\}^{\frac{1}{2}}\,o^{pt}_{(u)s;(v)t} = \sum_{j=1}^{A+1} \sum_{(w)} C^p_{t(u)s;j(w)}\,T_j\,o^p_{(w);(v)}, \quad (5·21)$$

showing that the numerical factor on the left-hand side is what is required to make the operator a normalized combination of coset operators. Since

$$C^p_{t(u)s;j(w)} = [f^{pt}/\{(A+1)f^p\}]^{\frac{1}{2}}\,\langle(u)\,s|\,T_j\,|(w)\,t\rangle, \quad (5·22)$$

it follows, from $T_{A+1} = I$, that

$$C^p_{t(u)s;A+1(w)} = [f^{pt}/\{(A+1)f^p\}]^{\frac{1}{2}}\,\delta((u),(w))\,\delta(s,t)$$
$$= c^p_{0t}\,\delta((u),(w))\,\delta(s,t), \quad (5·23)$$

with c^p_{0t} as in (2·6).

Thus, for $s \neq t$,
$$\{(A+1)f^p/f^{pt}\}^{\frac{1}{2}}\,o^{pt}_{(u)s;(v)t} = \sum_{j=1}^{A} \sum_{(w)} C^p_{t(u)s;j(w)}\,T_j\,o^p_{(w);(v)}, \quad (5·24)$$

whilst for $s = t$ we find

$$\{(A+1)f^p/f^{pt}\}^{\frac{1}{2}}\,o^{pt}_{(u)t;(v)t} = c^p_{0t}\,o^p_{(u);(v)} + \sum_{j=1}^{A} \sum_{(w)} C^p_{t(u)t;j(w)}\,T_j\,o^p_{(w);(v)}. \quad (5·25)$$

Comparison with (2·1) and (2·2) written in the form

$$\{(A+1)f^p/f^{pt}\}^{\frac{1}{2}}\,o^{pt}_{(u)s;(v)t} = o^{p't}_{(u);(v')t}\,(Af^p_s/f^{pt})^{\frac{1}{2}}\,P_{A,\,A+1}\,o^p_{(v')s;(v)} \quad (t \neq s), \quad (5·26)$$

$$\{(A+1)f^p/f^{pt}\}^{\frac{1}{2}}\,o^{pt}_{(u)t;(v)t} = c^p_{0t}\,o^p_{(u);(v)} + \sum_s c^p_{st}\,o^p_{(u);(v')s}\,(Af^p_s/f^p)^{\frac{1}{2}}\,P_{A,\,A+1}\,o^p_{(v')s;(v)}, \quad (5·27)$$

thus requires

$$o^{p't}_{(u);(v')t}\,(Af^p_s/f^{pt}_s)^{\frac{1}{2}}\,P_{A,\,A+1}\,o^p_{(v')s;(v)} = \sum_{j=1}^{A} \sum_{(w)} C^p_{t(u)s;j(w)}\,T_j\,o^p_{(w);(v)} \quad (s \neq t), \quad (5·28)$$

$$\sum_s c^p_{st}\,o^p_{(u);(v')s}\,(Af^p_s/f^p)^{\frac{1}{2}}\,P_{A,\,A+1}\,o^p_{(v')s;(v)} = \sum_{j=1}^{A} \sum_{(w)} C^p_{t(u)t;j(w)}\,T_j\,o^p_{(w);(v)}. \quad (5·29)$$

Multiplying both sides of (5·29) by $c^p_{s_1 t}$, summing over t and using the column orthogonality relation

$$\sum_t c^p_{st} c^p_{s_1 t} = \delta(s, s_1), \tag{5·30}$$

equation (5·29) becomes, with s_1 replaced again by s,

$$o^p_{(u);(v')s}(Af^p_s/f^p)^{\frac{1}{2}} P_{A,A+1} o^p_{(v')s;(v)} = \sum_{j=1}^{A} \sum_{(w)} k^p_{s(u);j(w)} T_j o^p_{(w);(v)}, \tag{5·31}$$

where

$$k^p_{s(u);j(w)} = \sum_t c^p_{st} C^p_{t(u)t;j(w)}. \tag{5·32}$$

Equations (5·28), (5·31) express the two-sided products of the Young operators of S_A with $P_{A,A+1}$ as linear combinations of proper (i.e. coset S_A not included) coset operators. We may show directly that the numerical factor $(Af^p_s/f^p)^{\frac{1}{2}}$ is what is required to normalize these combinations. This is of course already clear in the case of (5·28) since for $s \neq t$

$$\sum_{j=1}^{A} \sum_{(w)} (C^p_{t(u)s;j(w)})^2 = 1. \tag{5·33}$$

If we take, with $T_{A+1} = I$,

$$T_A = P_{A,A+1}, \tag{5·34}$$

so that the Ath coset is defined to contain $P_{A,A+1}$, then the coefficient of $P_{A,A+1} o^p_{(v')s;(v)}$ on the right-hand side of (5·28) is, with $s \neq t$,

$$\begin{aligned} C^p_{t(u)s;A(v')s} &= [f^{pt}/\{(A+1)f^p\}]^{\frac{1}{2}} \langle (u)\, s |\, P_{A,A+1} \,| (v')\, st \rangle \\ &= \delta((u),(v')\,t)\, [f^{pt}/\{(A+1)f^p\}]^{\frac{1}{2}} \langle (v')\, ts |\, P_{A,A+1} \,| (v')\, st \rangle \\ &= \delta((u),(v')\,t)\, [f^{pt}/\{(A+1)f^p\}]^{\frac{1}{2}} \{1-(r^{pt}_{st})^2\}^{\frac{1}{2}} \\ &= \delta((u),(v')\,t)\, [f^{pt}/\{(A+1)f^p\}]^{\frac{1}{2}} \{(A+1)f^{pt}_s f^p/(Af^{pt}f^p_s)\}^{\frac{1}{2}} \\ &= \delta((u),(v')\,t)\, \{f^{pt}_s/(Af^p_s)\}^{\frac{1}{2}}. \end{aligned} \tag{5·35}$$

Thus

$$c^{p't}_{(v')t;(v')t}(Af^p_s/f^{pt}_s)^{\frac{1}{2}} P_{A,A+1} o^p_{(v')s;(v)} = \{f^{pt}_s/(Af^p_s)\}^{\frac{1}{2}} P_{A,A+1} o^p_{(v')s;(v)} + \dots. \tag{5·36}$$

It follows, by Englefield's theorem (1956) on the normalization of expressions symmetrized by Young operators, that $(Af^p_s/f^{pt}_s)^{\frac{1}{2}}$ is the factor required to normalize

$$o^{p't}_{(v')t;(v')t} P_{A,A+1} o^p_{(v')s;(v)} \quad (s \neq t) \tag{5·37}$$

and that the *same* factor normalizes all the expressions

$$o^{p't}_{(u);(v')t} P_{A,A+1} o^p_{(v')s;(v)}. \tag{5·38}$$

In the case of (5·31) the coefficient of $P_{A,A+1} o^p_{(v')s;(v)}$ on the right-hand side is

$$\begin{aligned} k^p_{s(u);A(v')s} &= \sum_t c^p_{st} C^p_{t(u)t;A(v')s} \\ &= \sum_t c^p_{st} [f^{pt}/\{(A+1)f^p\}]^{\frac{1}{2}} \langle (u)\, t |\, P_{A,A+1} \,| (v')\, st \rangle \\ &= \delta((u),(v')\,s) \sum_t c^p_{st} r^p_{st} [f^{pt}/\{(A+1)f^p\}]^{\frac{1}{2}} \\ &= \delta((u),(v')\,s) \sum_t [Af^{pt}f^p_s/\{(A+1)\,(f^p)^2\}]^{\frac{1}{2}} (r^{pt}_{st})^2 [f^{pt}/\{(A+1)f^p\}]^{\frac{1}{2}} \\ &= \delta((u),(v')\,s)\, (Af^p_s/f^p)^{\frac{1}{2}} \{\sum_t f^{pt}(r^{pt}_{st})^2\}/\{(A+1)f^p\}. \end{aligned} \tag{5·39}$$

But from (2·44) we have

$$\sum_t f^{pt}(r^{pt}_{st})^2 = (A+1)\,(f^p)^2/(Af^p_s). \tag{5·40}$$

Thus
$$k^{p}_{s(u);\,A(v')s} = \delta((u),(v')\,s)\,\{f^{p}/(Af^{p}_{s})\}^{\frac{1}{2}}. \tag{5·41}$$

Hence
$$o^{p}_{(v')s;\,(v')s}\,(Af^{p}_{s}/f^{p})^{\frac{1}{2}}\,P_{A,\,A+1}\,o^{p}_{(v')s;\,(v)} = \{f^{p}/(Af^{p}_{s})\}^{\frac{1}{2}}\,P_{A,\,A+1}\,o^{p}_{(v')s;\,(v)} + \ldots, \tag{5·42}$$

showing again by Englefield's theorem, that the factor $(Af^{p}_{s}/f^{p})^{\frac{1}{2}}$ normalizes *all* the expressions
$$o^{p}_{(u);\,(v')s}\,P_{A,\,A+1}\,o^{p}_{(v')s;\,(v)}. \tag{5·43}$$

It follows
$$\sum_{j=1}^{A}\sum_{(w)}(k^{p}_{s(u);\,j(w)})^{2} = 1. \tag{5·44}$$

Equation (5·44), included in the more general relation
$$\sum_{j=1}^{A}\sum_{(w)} k^{p}_{s(u);\,j(w)}\,k_{s_1(u_1);\,j(w)} = \delta(s_1,s)\,\delta((u_1),(u)), \tag{5·45}$$

may be verified directly from (5·32), (5·14) and (5·23) using the orthogonality relations
$$\sum_{t} c^{p}_{st}c^{p}_{s_1t} = \delta(s_1,s), \quad \sum_{t} c^{p}_{0t}c^{p}_{st} = 0. \tag{5·46}$$

For we have
$$\begin{aligned}
\sum_{j=1}^{A}\sum_{(w)} k^{p}_{s(u);\,j(w)}k^{p}_{s_1(u_1);\,j(w)} &= \sum_{j=1}^{A}\sum_{(w)}\sum_{t}\sum_{t_1} c^{p}_{st}c^{p}_{s_1t_1}\,C^{p}_{t(u)t;\,j(w)}\,C^{p}_{t_1(u_1)t_1;\,j(w)}\\
&= \sum_{t}\sum_{t_1} c^{p}_{st}c^{p}_{s_1t_1}\,\{\delta(t_1,t)\,\delta((u_1),(u)) - \sum_{(w)} C^{p}_{t(u)t;\,A+1(w)}\,C^{p}_{t_1(u_1)t_1;\,A+1(w)}\}\\
&= \sum_{t}\sum_{t_1} c^{p}_{st}c^{p}_{s_1t_1}\,\{\delta(t_1,t) - c^{p}_{0t}c^{p}_{0t_1}\}\,\delta((u_1),(u))\\
&= \{\sum_{t} c^{p}_{st}c^{p}_{s_1t} - (\sum_{t} c^{p}_{0t}c^{p}_{st})^{2}\}\,\delta((u_1),(u))\\
&= \delta(s_1,s)\,\delta((u_1),(u)).
\end{aligned} \tag{5·47}$$

6. Explicit evaluation for the case of the Wigner nuclear supermultiplets

The permutation symmetry of the charge-spin states of an A-particle nucleus is resticted, by Pauli's principle, to representations $R^{\tilde{p}}$ of S_A characterized by partitions of A into not more than four parts
$$\tilde{p} = [\tilde{p}_1\tilde{p}_2\tilde{p}_3\tilde{p}_4], \quad \tilde{p}_1 \geqslant \tilde{p}_2 \geqslant \tilde{p}_3 \geqslant \tilde{p}_4 \geqslant 0, \quad \tilde{p}_1+\tilde{p}_2+\tilde{p}_3+\tilde{p}_4 = A. \tag{6·1}$$

This set of allowed partitions characterize the different Wigner (1937) supermultiplets. Such symmetrized nuclear charge-spin states may be combined only with orbital states of adjoint symmetry to form totally antisymmetric states. Thus the allowed nuclear orbital symmetry is described by the set of partitions
$$p = [4^{k_1}3^{k_2}2^{k_3}1^{k_4}], \quad 4k_1+3k_2+2k_3+k_4 = A, \tag{6·2}$$

where
$$k_1 = \tilde{p}_4, \quad k_2 = \tilde{p}_3-\tilde{p}_4, \quad k_3 = \tilde{p}_2-\tilde{p}_3, \quad k_4 = \tilde{p}_1-\tilde{p}_2, \tag{6·3}$$
$$\tilde{p}_1 = k_1+k_2+k_3+k_4 = k, \quad \tilde{p}_2 = k_1+k_2+k_3, \quad \tilde{p}_3 = k_1+k_2, \quad \tilde{p}_4 = k_1. \tag{6·4}$$

Since adjoint representations R^{p}, $R^{\tilde{p}}$ of S_A have the same dimension, we find, from the general dimension formula applied to $f^{\tilde{p}}$
$$f^{p} = f^{\tilde{p}} = A!\,D^{\tilde{p}}/(\tilde{m}_1!\,\tilde{m}_2!\,\tilde{m}_3!\,\tilde{m}_4!), \quad \tilde{m}_r = \tilde{p}_r+4-r, \tag{6·5}$$

where
$$D^{\tilde{p}} = D(\tilde{m}_1,\tilde{m}_2,\tilde{m}_3,\tilde{m}_4) = D(\tilde{p}_1+3,\tilde{p}_2+2,\tilde{p}_3+1,\tilde{p}_4). \tag{6·6}$$

It follows

$$f^p = \frac{A!\,(k_2+1)\,(k_3+1)\,(k_4+1)\,(k_2+k_3+2)\,(k_3+k_4+2)\,(k_2+k_3+k_4+3)}{k_1!\,(k_1+k_2+1)!\,(k_1+k_2+k_3+2)!\,(k_1+k_2+k_3+k_4+3)!}. \qquad (6\cdot7)$$

Since n, the number of distinct row lengths in the Young tableau of p, cannot exceed 4, we have at the most a 5-rowed square matrix of coefficients c^p_{0t}, c^p_{st}, where the row with $t = 1$, which will be given to complete the orthogonal matrix, will be of no physical interest. When the row number t of the added square has the value

$$t = k_1+k_2+\ldots+k_r+1 \quad (r = 1, 2, 3), \qquad (6\cdot8)$$

we obtain the dimension f^{pt} of the corresponding representation from formula (6·7) by means of the substitutions

$$A \to A+1, \quad k_r \to k_r+1, \quad k_{r+1} \to k_{r+1}-1 \quad (r = 1, 2, 3). \qquad (6\cdot9)$$

For the case

$$t = k_1+k_2+k_3+k_4+1 = k+1 \qquad (6\cdot10)$$

(added square at the bottom of the tableau) we substitute simply

$$A \to A+1, \quad k_4 \to k_4+1. \qquad (6\cdot11)$$

We find in this way the following formulae for the coefficients

$$c^p_{0t} = [f^{pt}/\{(A+1)f^p\}]^{\frac{1}{2}}, \quad t = k_1+k_2+\ldots+k_r+1 \quad (r = 1, 2, 3, 4). \qquad (6\cdot12)$$

$$c^p_{0,k_1+1} = \left\{\frac{k_2(k_2+k_3+1)\,(k_2+k_3+k_4+2)}{(k_1+1)\,(k_2+1)\,(k_2+k_3+2)\,(k_2+k_3+k_4+3)}\right\}^{\frac{1}{2}}, \qquad (6\cdot13)$$

$$c^p_{0,k_1+k_2+1} = \left\{\frac{k_3(k_2+2)\,(k_3+k_4+1)}{(k_2+1)\,(k_3+1)\,(k_1+k_2+2)\,(k_3+k_4+2)}\right\}^{\frac{1}{2}}, \qquad (6\cdot14)$$

$$c^p_{0,k-k_4+1} = \left\{\frac{k_4(k_3+2)\,(k_2+k_3+3)}{(k_3+1)\,(k_4+1)\,(k_2+k_3+2)\,(k_1+k_2+k_3+3)}\right\}^{\frac{1}{2}}, \qquad (6\cdot15)$$

$$c^p_{0,k+1} = \left\{\frac{(k_4+2)\,(k_3+k_4+3)\,(k_2+k_3+k_4+4)}{(k_4+1)\,(k_3+k_4+2)\,(k_2+k_3+k_4+3)\,(k_1+k_2+k_3+k_4+4)}\right\}^{\frac{1}{2}}. \qquad (6\cdot16)$$

The coefficient c^p_{01} required to complete the column of coefficients satisfying the normalization condition

$$\sum_t (c^p_{0t})^2 = 1, \qquad (6\cdot17)$$

involving an adjoint partition of A into five parts, may be calculated in a similar manner from the general dimension formula. We find

$$c^p_{01} = \left\{\frac{k_1(k_1+k_2+1)\,(k_1+k_2+k_3+2)\,(k_1+k_2+k_3+k_4+3)}{(k_1+1)\,(k_1+k_2+2)\,(k_1+k_2+k_3+3)\,(k_1+k_2+k_3+k_4+4)}\right\}^{\frac{1}{2}}. \qquad (6\cdot18)$$

From (2·7) we have

$$\begin{aligned} c^p_{st} &= r^{pt}_{st}\,[f^{pt}/\{(A+1)f^p\}]^{\frac{1}{2}}/\{f^p/(Af^p_s)\}^{\frac{1}{2}} \\ &= r^{pt}_{st}\,c^p_{0t}/c^{p/s}_{0s}, \end{aligned} \qquad (6\cdot19)$$

where

$$c^{p/s}_{0s} = \{f^p/(Af^p_s)\}^{\frac{1}{2}} \qquad (6\cdot20)$$

may be evaluated, for $s = k_1+k_2+\ldots+k_r$ $(r = 1,2,3)$ from

$$c^p_{0t} = [f^{pt}/\{(A+1)f^p\}]^{\frac{1}{2}}, \quad t = 1+k_1+k_2+\ldots+k_r \quad (r = 1,2,3) \tag{6·21}$$

by the substitution
$$k_r \to k_r-1, \quad k_{r+1} \to k_{r+1}+1, \tag{6·22}$$

and for $s = k$ from $c^p_{0,k+1}$ by the substitution

$$k_4 \to k_4-1. \tag{6·23}$$

Hence we deduce from (6·13) to (6·16) that

$$c^{p/k_1}_{0k_1} = \{f^p/(Af^p_{k_1})\}^{\frac{1}{2}} = \left\{\frac{(k_2+1)\,(k_2+k_3+2)\,(k_2+k_3+k_4+3)}{k_1(k_2+2)\,(k_2+k_3+3)\,(k_2+k_3+k_4+4)}\right\}^{\frac{1}{2}}, \tag{6·24}$$

$$c^{p/(k_1+k_2)}_{0,k_1+k_2} = \{f^p/(Af^p_{k_1+k_2})\}^{\frac{1}{2}} = \left\{\frac{(k_3+1)\,(k_2+1)\,(k_3+k_4+2)}{k_2(k_3+2)\,(k_1+k_2+1)\,(k_3+k_4+3)}\right\}^{\frac{1}{2}}, \tag{6·25}$$

$$c^{p/(k-k_4)}_{0,k-k_4} = \{f^p/(Af^p_{k-k_4})\}^{\frac{1}{2}} = \left\{\frac{(k_4+1)\,(k_3+1)\,(k_2+k_3+2)}{k_3(k_4+2)\,(k_2+k_3+1)\,(k_1+k_2+k_3+2)}\right\}^{\frac{1}{2}}, \tag{6·26}$$

$$c^{p/k}_{0k} = \{f^p/(Af^p_k)\}^{\frac{1}{2}} = \left\{\frac{(k_4+1)\,(k_3+k_4+2)\,(k_2+k_3+k_4+3)}{k_4(k_3+k_4+1)\,(k_2+k_3+k_4+2)\,(k_1+k_2+k_3+k_4+3)}\right\}^{\frac{1}{2}}. \tag{6·27}$$

The reciprocal, r^{pt}_{st}, of the Young axial distance, occurring in (6·19), is easily deduced from the corresponding Young tableaus. We give below the 5×4 matrix of Young axial distances:

$$a^{pt}_{st} = 1/r^{pt}_{st} = p_t+1-p_s+s-t. \tag{6·28}$$

TABLE OF AXIAL DISTANCES a^{pt}_{st} FOR $p = [4^{k_1}3^{k_2}2^{k_3}1^{k_4}]$ $(k = k_1+k_2+k_3+k_4)$ (6·29)

	$s = k_1$	$s = k_1+k_2$	$s = k-k_4$	$s = k$
$t = 1$	k_1	k_1+k_2+1	$k-k_4+2$	$k+3$
$t = 1+k_1$	-1	k_2	k_2+k_3+1	$k-k_1+2$
$t = 1+k_1+k_2$	$-(k_2+2)$	-1	k_3	k_3+k_4+1
$t = 1+k-k_4$	$-(k_2+k_3+3)$	$-(k_3+2)$	-1	k_4
$t = 1+k$	$-(k-k_1+4)$	$-(k_3+k_4+3)$	$-(k_4+2)$	-1

When we evaluate c^p_{st} from (6·19) we find that the factor from a^{pt}_{st} always cancels, leaving only the sign. We find in this way the complete set of c^p_{st} coefficients for $p = [4^{k_1}3^{k_2}2^{k_3}1^{k_4}]$ given below, where again we have put $k = k_1+k_2+k_3+k_4$.

$\underline{t = 1}$

$\underline{s = k_1}$
$$c^p_{k_11} = \left\{\frac{(k_2+2)\,(k_1+k_2+1)\,(k_2+k_3+3)\,(k-k_4+2)\,(k-k_1+4)\,(k+3)}{(k_1+1)\,(k_2+1)\,(k_1+k_2+2)\,(k_2+k_3+2)\,(k-k_4+3)\,(k-k_1+3)\,(k+4)}\right\}^{\frac{1}{2}}, \tag{6·30}$$

$\underline{s = k_1+k_2}$
$$c^p_{k_1+k_2,1} = \left\{\frac{k_1k_2(k_3+2)\,(k_3+k_4+3)\,(k-k_4+2)\,(k+3)}{(k_1+1)\,(k_2+1)\,(k_3+1)\,(k_1+k_2+2)\,(k_3+k_4+2)\,(k-k_4+3)\,(k+4)}\right\}^{\frac{1}{2}}, \tag{6·31}$$

$\underline{s = k-k_4}$
$$c^p_{k-k_4,1} = \left\{\frac{k_1k_3(k_4+2)\,(k_1+k_2+1)\,(k_2+k_3+1)\,(k+3)}{(k_1+1)\,(k_3+1)\,(k_4+1)\,(k_1+k_2+2)\,(k_2+k_3+2)\,(k-k_4+3)\,(k+4)}\right\}^{\frac{1}{2}}, \tag{6·32}$$

$\underline{s = k}$
$$c^p_{k1} = \left\{\frac{k_1k_4(k_1+k_2+1)\,(k_3+k_4+1)\,(k-k_4+2)\,(k-k_1+2)}{(k_1+1)\,(k_4+1)\,(k_1+k_2+2)\,(k_3+k_4+2)\,(k-k_4+3)\,(k-k_1+3)\,(k+4)}\right\}^{\frac{1}{2}}, \tag{6·33}$$

$t = 1+k_1$

$s = k_1$ $$c^p_{k_1,\,1+k_1} = -\left\{\frac{k_1k_2(k_2+2)\,(k_2+k_3+1)\,(k_2+k_3+3)\,(k-k_1+2)\,(k-k_1+4)}{(k_1+1)\,(k_2+1)^2\,(k_2+k_3+2)^2\,(k-k_1+3)^2}\right\}^{\frac{1}{2}}, \quad (6\cdot34)$$

$s = k_1+k_2$ $$c^p_{k_1+k_2,\,1+k_1} = \left\{\frac{(k_3+2)\,(k_1+k_2+1)\,(k_2+k_3+1)\,(k_3+k_4+3)\,(k-k_1+2)}{(k_1+1)\,(k_2+1)^2\,(k_3+1)\,(k_2+k_3+2)\,(k_3+k_4+2)\,(k-k_1+3)}\right\}^{\frac{1}{2}}, \quad (6\cdot35)$$

$s = k-k_4$ $$c^p_{k-k_4,\,1+k_1} = \left\{\frac{k_2k_3(k_4+2)\,(k-k_1+2)\,(k-k_4+2)}{(k_1+1)\,(k_2+1)\,(k_3+1)\,(k_4+1)\,(k_2+k_3+2)^2\,(k-k_1+3)}\right\}^{\frac{1}{2}}, \quad (6\cdot36)$$

$s = k$ $$c^p_{k,\,1+k_1} = \left\{\frac{k_2k_4(k_2+k_3+1)\,(k_3+k_4+1)\,(k+3)}{(k_1+1)\,(k_2+1)\,(k_4+1)\,(k_2+k_3+2)\,(k_3+k_4+2)\,(k-k_1+3)^2}\right\}^{\frac{1}{2}}, \quad (6\cdot37)$$

$t = 1+k_1+k_2$

$s = k_1$ $$c^p_{k_1,\,1+k_1+k_2} = -\left\{\frac{k_1k_3(k_2+k_3+3)\,(k_3+k_4+1)\,(k-k_1+4)}{(k_2+1)^2\,(k_3+1)\,(k_1+k_2+2)\,(k_2+k_3+2)\,(k_3+k_4+2)\,(k-k_1+3)}\right\}^{\frac{1}{2}}, \quad (6\cdot38)$$

$s = k_1+k_2$ $$c^p_{k_1+k_2,\,1+k_1+k_2} = -\left\{\frac{k_2k_3(k_2+2)\,(k_3+2)\,(k_1+k_2+1)\,(k_3+k_4+1)\,(k_3+k_4+3)}{(k_2+1)^2\,(k_3+1)^2\,(k_1+k_2+2)\,(k_3+k_4+2)^2}\right\}^{\frac{1}{2}}, \quad (6\cdot39)$$

$s = k-k_4$ $$c^p_{k-k_4,\,1+k_1+k_2} = \left\{\frac{(k_2+2)\,(k_4+2)\,(k_2+k_3+1)\,(k_3+k_4+1)\,(k-k_4+2)}{(k_2+1)\,(k_3+1)^2\,(k_4+1)\,(k_1+k_2+2)\,(k_2+k_3+2)\,(k_3+k_4+2)}\right\}^{\frac{1}{2}}, \quad (6\cdot40)$$

$s = k$ $$c^p_{k,\,1+k_1+k_2} = \left\{\frac{k_3k_4(k_2+2)\,(k-k_1+2)\,(k+3)}{(k_2+1)\,(k_3+1)\,(k_4+1)\,(k_1+k_2+2)\,(k_3+k_4+2)^2\,(k-k_1+3)}\right\}^{\frac{1}{2}}, \quad (6\cdot41)$$

$t = 1+k-k_4$

$s = k_1$ $$c^p_{k_1,\,1+k-k_4} = -\left\{\frac{k_1k_4(k_2+2)\,(k_3+2)\,(k-k_1+4)}{(k_2+1)\,(k_3+1)\,(k_4+1)\,(k_2+k_3+2)^2\,(k-k_4+3)\,(k-k_1+3)}\right\}^{\frac{1}{2}}, \quad (6\cdot42)$$

$s = k_1+k_2$ $$c^p_{k_1+k_2,\,1+k-k_4} = -\left\{\frac{k_2k_4(k_1+k_2+1)\,(k_2+k_3+3)\,(k_3+k_4+3)}{(k_2+1)\,(k_3+1)^2\,(k_4+1)\,(k_2+k_3+2)\,(k_3+k_4+2)\,(k-k_4+3)}\right\}^{\frac{1}{2}}, \quad (6\cdot43)$$

$s = k-k_4$ $$c^p_{k-k_4,\,1+k-k_4} = -\left\{\frac{k_3k_4(k_3+2)\,(k_4+2)\,(k_2+k_3+1)\,(k_2+k_3+3)\,(k-k_4+2)}{(k_3+1)^2\,(k_4+1)^2\,(k_2+k_3+2)^2\,(k-k_4+3)}\right\}^{\frac{1}{2}}, \quad (6\cdot44)$$

$s = k$ $$c^p_{k,\,1+k-k_4} = \left\{\frac{(k_3+2)\,(k_2+k_3+3)\,(k_3+k_4+1)\,(k-k_1+2)\,(k+3)}{(k_3+1)\,(k_4+1)^2\,(k_2+k_3+2)\,(k_3+k_4+2)\,(k-k_4+3)\,(k-k_1+3)}\right\}^{\frac{1}{2}}, \quad (6\cdot45)$$

$t = 1+k$

$s = k_1$ $$c^p_{k_1,\,1+k} = -\left\{\frac{k_1(k_2+2)\,(k_4+2)\,(k_2+k_3+3)\,(k_3+k_4+3)}{(k_2+1)\,(k_4+1)\,(k_2+k_3+2)\,(k_3+k_4+2)\,(k-k_1+3)^2\,(k+4)}\right\}^{\frac{1}{2}}, \quad (6\cdot46)$$

$s = k_1+k_2$ $$c^p_{k_1+k_2,\,1+k} = -\left\{\frac{k_2(k_3+2)\,(k_4+2)\,(k_1+k_2+1)\,(k-k_1+4)}{(k_2+1)\,(k_3+1)\,(k_4+1)\,(k_3+k_4+2)^2\,(k-k_1+3)\,(k+4)}\right\}^{\frac{1}{2}}, \quad (6\cdot47)$$

$s = k-k_4$ $$c^p_{k-k_4,\,1+k} = -\left\{\frac{k_3(k_2+k_3+1)\,(k_3+k_4+3)\,(k-k_4+2)\,(k-k_1+4)}{(k_3+1)\,(k_4+1)^2\,(k_2+k_3+2)\,(k_3+k_4+2)\,(k-k_1+3)\,(k+4)}\right\}^{\frac{1}{2}}, \quad (6\cdot48)$$

$s = k$ $$c^p_{k,\,1+k} = -\left\{\frac{k_4(k_4+2)\,(k_3+k_4+1)\,(k_3+k_4+3)\,(k-k_1+2)\,(k-k_1+4)\,(k+3)}{(k_4+1)^2\,(k_3+k_4+2)^2\,(k-k_1+3)^2\,(k+4)}\right\}^{\frac{1}{2}}. \quad (6\cdot49)$$

7. Complete set of Young operators for S_3, S_4, S_5 and S_6: relations for adjoint partitions

The basic equations (2·1), (2·2) give rise to a recursive construction of the Young operators of S_{A+1} in terms of those of S_A which is much simpler than that given, for example, by Thrall (1941), since only the transpositions $P_{12}, P_{23}, \ldots, P_{A,A+1}$ need be considered. We illustrate this and the evaluation of the general formulae of § 6 by an explicit construction of all the Young operators for S_3, S_4, S_5 and S_6.

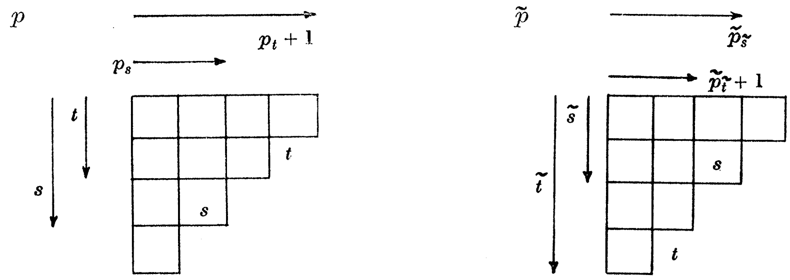
Before we proceed to this tabulation we note first, however, that there is a relation between the coefficients c^p_{0t}, c^p_{st} for the partition p and those for the partition $\tilde{p}$ adjoint to p. This relation shortens the tabulation and in the case when p is self-adjoint, $p = \tilde{p}$, leads to identities between the coefficients of the single partition. If p is a partition, s the row number of a square labelled s in p and t the row number of the added square labelled t, then by reflecting the Young tableau along the main diagonal, so that p is converted into $\tilde{p}$, we see (cf. the diagram below) that the same relative positions of s and t in $\tilde{p}$ are achieved by putting

$$\tilde{t} = p_t + 1, \quad \tilde{s} = p_s, \tag{7·1}$$

$$\tilde{p}_{\tilde{t}} = t - 1, \quad \tilde{p}_{\tilde{s}} = s. \tag{7·2}$$

By this reflexion, however, the axial distance from t to s changes sign. This may be verified by direct calculation. The axial distance from t to s in $\tilde{p}$ is given by

$$\begin{aligned} a^{\tilde{p}\tilde{t}}_{\tilde{s}\tilde{t}} &= \tilde{p}_{\tilde{t}} + 1 - \tilde{p}_{\tilde{s}} + \tilde{s} - \tilde{t} \\ &= (t-1) + 1 - s + p_s - (p_t + 1) \\ &= -(p_t + 1 - p_s + s - t) \\ &= -a^{pt}_{st}. \end{aligned} \tag{7·3}$$



Since the dimensions of adjoint representations are equal we have then

$$f^{\tilde{p}} = f^p, \quad f^{\tilde{p}\tilde{t}} = f^{pt}, \quad f^{\tilde{p}}_{\tilde{s}} = f^p_s. \tag{7·4}$$

It follows, from (2·6), (2·7) that

$$c^{\tilde{p}}_{0\tilde{t}} = c^p_{0t}, \quad c^{\tilde{p}}_{\tilde{s}\tilde{t}} = -c^p_{st}, \tag{7·5}$$

which are the relations in question, holding also when $\tilde{p} = p$.

The above relations have been used below in the tabulation of c^p_{0t}, c^p_{st} for $A+1 = 3$, 4, 5 and 6. The algebraic labels have been retained for s and t to make clear the specialization required in applying the general formulae of § 6 to cases where some of the exponents k_1, k_2, k_3 and k_4 are zero. The general formulae have been checked in this way although, of course, for small A, where the dimensions of the representations are well known, equations (2·6) and (2·7) are easy to evaluate directly.

$\underline{A+1=3}$

$p=[2],\quad k_3=1,\quad k_1=k_2=k_4=0,\quad k=1.$

$p=[1^2],\quad k_4=2,\quad k_1=k_2=k_3=0,\quad k=2.$

$$\begin{aligned} &c^{[2]}_{0,k_1+k_2+1}=(\tfrac{1}{3})^{\frac{1}{2}}=c^{[1^2]}_{0,k+1}; && c^{[2]}_{k-k_4,k_1+k_2+1}=(\tfrac{2}{3})^{\frac{1}{2}}=-c^{[1^2]}_{k,k+1};\\ &c^{[2]}_{0,k+1}=(\tfrac{2}{3})^{\frac{1}{2}}=c^{[1^2]}_{0,k-k_4+1}; && c^{[2]}_{k-k_4,k+1}=-(\tfrac{1}{3})^{\frac{1}{2}}=-c^{[1^2]}_{k,k-k_4+1}. \end{aligned} \tag{7·6}$$

$\underline{A+1=4}$

$p=[3],\quad k_2=1,\quad k_1=k_3=k_4=0,\quad k=1.$

$\tilde{p}=[1^3],\quad k_4=3,\quad k_1=k_2=k_3=0,\quad k=3.$

$$\begin{aligned} &c^{[3]}_{0,k_1+1}=(\tfrac{1}{4})^{\frac{1}{2}}=c^{[1^3]}_{0,k+1}; && c^{[3]}_{k_1+k_2,k_1+1}=(\tfrac{3}{4})^{\frac{1}{2}}=-c^{[1^3]}_{k,k+1};\\ &c^{[3]}_{0,k+1}=(\tfrac{3}{4})^{\frac{1}{2}}=c^{[1^3]}_{0,k-k_4+1}; && c^{[3]}_{k_1+k_2,k+1}=-(\tfrac{1}{4})^{\frac{1}{2}}=-c^{[1^3]}_{k,k-k_4+1}. \end{aligned} \tag{7·7}$$

$p=[21]=\tilde{p},\quad k_3=k_4=1,\quad k_1=k_2=0,\quad k=2.$

$$\begin{aligned} &c^{[21]}_{0,k_1+k_2+1}=(\tfrac{6}{16})^{\frac{1}{2}}=c^{[21]}_{0,k+1}; && c^{[21]}_{k-k_4,k_1+k_2+1}=(\tfrac{9}{16})^{\frac{1}{2}}=-c^{[21]}_{k,k+1};\\ &c^{[21]}_{0,k-k_4+1}=(\tfrac{4}{16})^{\frac{1}{2}}; && c^{[21]}_{k-k_4,k-k_4+1}=-(\tfrac{6}{16})^{\frac{1}{2}}=-c^{[21]}_{k,k-k_4+1};\\ & && c^{[21]}_{k-k_4,k+1}=-(\tfrac{1}{16})^{\frac{1}{2}}=-c^{[21]}_{k,k_1+k_2+1}. \end{aligned} \tag{7·8}$$

$\underline{A+1=5}$

$p=[4],\quad k_1=1,\quad k_2=k_3=k_4=0,\quad k=1.$

$\tilde{p}=[1^4],\quad k_4=4,\quad k_1=k_2=k_3=0,\quad k=4.$

$$\begin{aligned} &c^{[4]}_{01}=(\tfrac{1}{5})^{\frac{1}{2}}=c^{[1^4]}_{0,k+1}; && c^{[4]}_{k_1,1}=(\tfrac{4}{5})^{\frac{1}{2}}=-c^{[1^4]}_{k,k+1};\\ &c^{[4]}_{0,k+1}=(\tfrac{4}{5})^{\frac{1}{2}}=c^{[1^4]}_{0,k-k_4+1}; && c^{[4]}_{k_1,k+1}=-(\tfrac{1}{5})^{\frac{1}{2}}=-c^{[1^4]}_{k,k-k_4+1}. \end{aligned} \tag{7·9}$$

$p=[31],\quad k_2=k_4=1,\quad k_1=k_3=0,\quad k=2.$

$\tilde{p}=[21^2],\quad k_3=1,\quad k_4=2,\quad k_1=k_2=0,\quad k=3.$

$$\begin{aligned} &c^{[31]}_{0,k_1+1}=(\tfrac{12}{45})^{\frac{1}{2}}=c^{[21^2]}_{0,k+1}; && c^{[31]}_{k_1+k_2,k_1+1}=(\tfrac{32}{45})^{\frac{1}{2}}=-c^{[21^2]}_{k,k+1};\\ &c^{[31]}_{0,k-k_4+1}=(\tfrac{15}{45})^{\frac{1}{2}}=c^{[21^2]}_{0,k-k_4+1}; && c^{[31]}_{k_1+k_2,k-k_4+1}=-(\tfrac{10}{45})^{\frac{1}{2}}=-c^{[21^2]}_{k,k-k_4+1};\\ &c^{[31]}_{0,k+1}=(\tfrac{18}{45})^{\frac{1}{2}}=c^{[21^2]}_{0,k_1+k_2+1}; && c^{[31]}_{k_1+k_2,k+1}=-(\tfrac{3}{45})^{\frac{1}{2}}=-c^{[21^2]}_{k,k_1+k_2+1};\\ &c^{[31]}_{k,k_1+1}=(\tfrac{1}{45})^{\frac{1}{2}}=-c^{[21^2]}_{k-k_4,k+1};\\ &c^{[31]}_{k,k-k_4+1}=(\tfrac{20}{45})^{\frac{1}{2}}=-c^{[21^2]}_{k-k_4,k-k_4+1};\\ &c^{[31]}_{k,k+1}=-(\tfrac{24}{45})^{\frac{1}{2}}=-c^{[21^2]}_{k-k_4,k_1+k_2+1}. \end{aligned} \tag{7·10}$$

$p=[2^2]=\tilde{p},\quad k_3=2,\quad k_1=k_2=k_4=0,\quad k=2.$

$$c^{[2^2]}_{0,k_1+k_2+1}=(\tfrac{1}{2})^{\frac{1}{2}}=c^{[2^2]}_{0,k+1};\quad c^{[2^2]}_{k-k_4,k_1+k_2+1}=(\tfrac{1}{2})^{\frac{1}{2}}=-c^{[2^2]}_{k-k_4,k+1}. \tag{7·11}$$

$\underline{A+1=6}$

$p=[41],\quad k_1=k_4=1,\quad k_2=k_3=0,\quad k=2.$

$\tilde{p}=[21^3],\quad k_3=1, k_4=3,\quad k_1=k_2=0,\quad k=4.$

$$\begin{aligned} &c^{[41]}_{01}=(\tfrac{20}{96})^{\frac{1}{2}}=c^{[21^2]}_{0,k+1}; && c^{[41]}_{k_1 1}=(\tfrac{75}{96})^{\frac{1}{2}}=-c^{[21^3]}_{k,k+1};\\ &c^{[41]}_{0,k-k_4+1}=(\tfrac{36}{96})^{\frac{1}{2}}=c^{[21^3]}_{0,k-k_4+1}; && c^{[41]}_{k_1,k-k_4+1}=-(\tfrac{15}{96})^{\frac{1}{2}}=-c^{[21^3]}_{k,k-k_4+1};\\ &c^{[41]}_{0,k+1}=(\tfrac{40}{96})^{\frac{1}{2}}=c^{[21^3]}_{0,k_1+k_2+1}; && c^{[41]}_{k_1,k+1}=-(\tfrac{6}{96})^{\frac{1}{2}}=-c^{[21^3]}_{k,k_1+k_2+1};\\ &c^{[41]}_{k1}=(\tfrac{1}{96})^{\frac{1}{2}}=-c^{[21^3]}_{k-k_4,k+1};\\ &c^{[41]}_{k,k-k_4+1}=(\tfrac{45}{96})^{\frac{1}{2}}=-c^{[21^3]}_{k-k_4,k-k_4+1};\\ &c^{[41]}_{k,k+1}=-(\tfrac{50}{96})^{\frac{1}{2}}=-c^{[21^3]}_{k-k_4,k_1+k_2+1}. \end{aligned} \tag{7·12}$$

$p = [32], \quad k_2 = k_3 = 1, \quad k_1 = k_4 = 0, \quad k = 2.$

$p = [2^21], \quad k_3 = 2, \quad k_4 = 1, \quad k_1 = k_2 = 0, \quad k = 3.$

$$c^{[32]}_{0,k_1+1} = (\tfrac{9}{30})^{\frac{1}{2}} = c^{[2^21]}_{0,k+1}; \qquad c^{[32]}_{k_1+k_2,k_1+1} = (\tfrac{18}{30})^{\frac{1}{2}} = -c^{[2^21]}_{k,k+1};$$
$$c^{[32]}_{0,k_1+k_2+1} = (\tfrac{5}{30})^{\frac{1}{2}} = c^{[2^21]}_{0,k-k_4+1}; \qquad c^{[32]}_{k_1+k_2,k_1+k_2+1} = -(\tfrac{10}{30})^{\frac{1}{2}} = -c^{[2^21]}_{k,k-k_4+1};$$
$$c^{[32]}_{0,k+1} = (\tfrac{16}{30})^{\frac{1}{2}} = c^{[2^21]}_{0,k_1+k_2+1}; \qquad c^{[32]}_{k_1+k_2,k+1} = -(\tfrac{2}{30})^{\frac{1}{2}} = -c^{[2^21]}_{k,k_1+k_2+1};$$
$$c^{[32]}_{k-k_4,k_1+1} = (\tfrac{3}{30})^{\frac{1}{2}} = -c^{[2^21]}_{k-k_4,k+1};$$
$$c^{[32]}_{k-k_4,k_1+k_2+1} = (\tfrac{15}{30})^{\frac{1}{2}} = -c^{[2^21]}_{k-k_4,k-k_4+1};$$
$$c^{[32]}_{k-k_4,k+1} = -(\tfrac{12}{30})^{\frac{1}{2}} = -c^{[2^21]}_{k-k_4,k_1+k_2+1}. \tag{7·13}$$

$p = [31^2] = \tilde{p}, \quad k_2 = 1, \quad k_4 = 2, \quad k_1 = k_3 = 0, \quad k = 3.$

$$c^{[31^2]}_{0,k_1+1} = (\tfrac{10}{36})^{\frac{1}{2}} = c^{[31^2]}_{0,k+1}; \qquad c^{[31^2]}_{k_1+k_2,k_1+1} = (\tfrac{25}{36})^{\frac{1}{2}} = -c^{[31^2]}_{k,k+1};$$
$$c^{[31^2]}_{0,k-k_4+1} = (\tfrac{16}{36})^{\frac{1}{2}}; \qquad c^{[31^2]}_{k_1+k_2,k-k_4+1} = -(\tfrac{10}{36})^{\frac{1}{2}} = -c^{[31^2]}_{k,k-k_4+1};$$
$$c^{[31^2]}_{k_1+k_2,k+1} = -(\tfrac{1}{36})^{\frac{1}{2}} = -c^{[31^2]}_{k,k_1+1}. \tag{7·14}$$

For completeness we add the following, deduced easily from (2·6), (2·7):

$$\left.\begin{aligned} c^{[5]}_{01} &= (\tfrac{1}{6})^{\frac{1}{2}} = c^{[1^5]}_{06}; \quad c^{[5]}_{11} = (\tfrac{5}{6})^{\frac{1}{2}} = -c^{[1^5]}_{56}, \\ c^{[5]}_{02} &= (\tfrac{5}{6})^{\frac{1}{2}} = c^{[1^5]}_{01}; \quad c^{[5]}_{12} = -(\tfrac{1}{6})^{\frac{1}{2}} = -c^{[1^5]}_{51}. \end{aligned}\right\} \tag{7·15}$$

From the above list of coefficients c^p_{0t}, c^p_{st}, inserted into (2·5) together with the relation (2·1) we construct below a complete set of Young operators for S_3, S_4, S_5 and S_6. In tabulating (2·5) we have introduced in the heading of the table, for ease of presentation, the abbreviations

$$\langle p; p\rangle \equiv o^p_{(u);(v)}, \tag{7·16}$$

$$\left\langle \begin{matrix} (Af^p_s/f^p)^{\frac{1}{2}} P_{A,A+1} \\ p;\, p's;\, p \end{matrix} \right\rangle \equiv o^p_{(u);(v')s} (Af^p_s/f^p)^{\frac{1}{2}} P_{A,A+1}\, o^p_{(v')s;(v)}, \tag{7·17}$$

so that the partition p of A on the left of the symbol stands for any Yamanouchi symbol (u) of the representation R^p of S_A, whilst the same partition p on the right of the symbol stands for any other (and possibly the same) Yamanouchi symbol (v) of the same representation. The partition p' of $A-1$ stands for *any* Yamanouchi symbol (v') of $R^{p'} = R^p_s$. With respect to the numerical factors $(Af^p_s/f^p)^{\frac{1}{2}}$ occurring in the headings we may note that the sum of the squares of these for a given table is equal to A as a consequence of (1·11)

$$\sum_s (Af^p_s/f^p) = A. \tag{7·18}$$

Similarly in the row labels of the table we use the abbreviation

$$o^{pt}_{pt;pt} \equiv o^{pt}_{(u)t;(v)t}, \tag{7·19}$$

it being implied that the same values (u) and (v) are substituted for p on the left and right of each symbol in the heading for the row in question. The tabulation has been made more concise also by use of relation (7·5) which connects Young operators associated with adjoint partitions, the labels at the bottom of the table to be read in conjunction with the row labels on the right-hand side of the table. The minus sign arising from (7·5) is incorporated in the labels at the bottom of the table.

In tabulating (2·1), or (5·26), we use similarly the abbreviations

$$\left\langle \begin{matrix} (Af^{p'}/f^q)^{\frac{1}{2}} P_{A,A+1} \\ q;\, tp's;\, p \end{matrix} \right\rangle \equiv o^{p't}_{(u);(v')t} (Af^{p'}/f^q)^{\frac{1}{2}} P_{A,A+1}\, o^{p's}_{(v')s;(v)}, \tag{7·20}$$

$$o^{pt}_{qs;pt} \equiv o^{pt}_{(u)s;(v)t}. \tag{7·21}$$

Thus here the partition q of A stands for any Yamanouchi symbol (u) of the representation $R^q = R^{p't} = R^{pt}_s$ of S_A, the partition p' of $A-1$ for the Yamanouchi symbol (v') of the representation $R^{p'} = R^p_s = R^q_t$ of S_{A-1}, the partition p of A for the Yamanouchi symbol (v) of the representation $R^p = R^{p's} = R^{qs}_t$ of S_A; the so constructed Young operator belongs to the representation

$$R^{p'st} = R^{pt} = R^{qs} \quad (s \neq t) \tag{7·22}$$

of S_{A+1}.

Young operators for S_2

$$o^{[2]}_{[2];[2]} \equiv o^{[2]}_{11;11} = \tfrac{1}{2}(I+P_{12})\,;\; o^{[1^2]}_{[1^2];[1^2]} \equiv o^{[1^2]}_{12;12} = \tfrac{1}{2}(I-P_{12}). \tag{7·23}$$

Young operators for S_3

$p = [2]$

	$\langle[2];\,[2]\rangle$	$\left\langle \genfrac{}{}{0pt}{}{2^{\frac{1}{2}}P_{23}}{[2];\,[1]\,1;\,[2]} \right\rangle$	
$(\frac{3}{1})^{\frac{1}{2}}\, o^{[3]}_{[2]1;[2]1}$	$(\frac{1}{3})^{\frac{1}{2}}$	$(\frac{2}{3})^{\frac{1}{2}}$	$(\frac{3}{1})^{\frac{1}{2}}\, o^{[1^3]}_{[1^2]3;[1^2]3}$
$(\frac{3}{2})^{\frac{1}{2}}\, o^{[21]}_{[2]2;[2]2}$	$(\frac{2}{3})^{\frac{1}{2}}$	$-(\frac{1}{3})^{\frac{1}{2}}$	$(\frac{3}{2})^{\frac{1}{2}}\, o^{[21]}_{[1^2]1;[1^2]1}$
	$\langle[1^2];\,[1^2]\rangle$	$\left\langle \genfrac{}{}{0pt}{}{-2^{\frac{1}{2}}P_{23}}{[1^2];\,[1]\,2;\,[1^2]} \right\rangle$	$p = [1^2]$.

(7·24)

$p' = [1]$

$$(\tfrac{3}{2})^{\frac{1}{2}}\, o^{[21]}_{[2]2;[1^2]1} = \left\langle \genfrac{}{}{0pt}{}{2^{\frac{1}{2}}P_{23}}{[2];\,1\,[1]\,2;\,[1^2]} \right\rangle,$$

$$(\tfrac{3}{2})^{\frac{1}{2}}\, o^{[21]}_{[1^2]1;[2]2} = \left\langle \genfrac{}{}{0pt}{}{2^{\frac{1}{2}}P_{23}}{[1^2];\,2\,[1]\,1;\,[2]} \right\rangle. \tag{7·25}$$

Young operators for S_4

$p = [3]$

	$\langle[3];\,[3]\rangle$	$\left\langle \genfrac{}{}{0pt}{}{3^{\frac{1}{2}}P_{34}}{[3];\,[2]\,1;\,[3]} \right\rangle$	
$(\frac{4}{1})^{\frac{1}{2}}\, o^{[4]}_{[3]1;[3]1}$	$(\frac{1}{4})^{\frac{1}{2}}$	$(\frac{3}{4})^{\frac{1}{2}}$	$(\frac{4}{1})^{\frac{1}{2}}\, o^{[1^4]}_{[1^3]4;[1^3]4}$
$(\frac{4}{3})^{\frac{1}{2}}\, o^{[31]}_{[3]2;[3]2}$	$(\frac{3}{4})^{\frac{1}{2}}$	$-(\frac{1}{4})^{\frac{1}{2}}$	$(\frac{4}{3})^{\frac{1}{2}}\, o^{[21^2]}_{[1^3]1;[1^3]1}$
	$\langle[1^3];\,[1^3]\rangle$	$\left\langle \genfrac{}{}{0pt}{}{-3^{\frac{1}{2}}P_{34}}{[1^3];\,[1^2]\,3;\,[1^3]} \right\rangle$	$p = [1^3]$.

(7·26)

$p = [21]$

	$\langle[21];\,[21]\rangle$	$\left\langle \genfrac{}{}{0pt}{}{(\frac{3}{2})^{\frac{1}{2}}P_{34}}{[21];\,[1^2]\,1;\,[21]} \right\rangle$	$\left\langle \genfrac{}{}{0pt}{}{(\frac{3}{2})^{\frac{1}{2}}P_{34}}{[21];\,[2]\,2;\,[21]} \right\rangle$
$(\frac{8}{3})^{\frac{1}{2}}\, o^{[31]}_{[21]1;[21]1}$	$(\frac{6}{16})^{\frac{1}{2}}$	$(\frac{9}{16})^{\frac{1}{2}}$	$(\frac{1}{16})^{\frac{1}{2}}$
$(\frac{8}{2})^{\frac{1}{2}}\, o^{[2^2]}_{[21]2;[21]2}$	$(\frac{4}{16})^{\frac{1}{2}}$	$-(\frac{6}{16})^{\frac{1}{2}}$	$(\frac{6}{16})^{\frac{1}{2}}$
$(\frac{8}{3})^{\frac{1}{2}}\, o^{[21^2]}_{[21]3;[21]2}$	$(\frac{6}{16})^{\frac{1}{2}}$	$-(\frac{1}{16})^{\frac{1}{2}}$	$-(\frac{9}{16})^{\frac{1}{2}}$

(7·27)

$p' = [2]$ $\qquad$ $p' = [1^2]$

$$(\tfrac{8}{3})^{\frac{1}{2}}\, o^{[31]}_{[3]2;[21]1} = \left\langle \genfrac{}{}{0pt}{}{3^{\frac{1}{2}}P_{34}}{[3];\,1\,[2]\,2;\,[21]} \right\rangle; \quad (\tfrac{8}{3})^{\frac{1}{2}}\, o^{[21^2]}_{[1^3]1;[21]3} = \left\langle \genfrac{}{}{0pt}{}{3^{\frac{1}{2}}P_{34}}{[1^3];\,3\,[1^2]\,1;\,[21]} \right\rangle;$$

$$(\tfrac{4}{3})^{\frac{1}{2}}\, o^{[31]}_{[21]1;[3]2} = \left\langle \genfrac{}{}{0pt}{}{(\frac{3}{2})^{\frac{1}{2}}P_{34}}{[21];\,2\,[2]\,1;\,[3]} \right\rangle; \quad (\tfrac{4}{3})^{\frac{1}{2}}\, o^{[21^2]}_{[21]3;[1^3]1} = \left\langle \genfrac{}{}{0pt}{}{(\frac{3}{2})^{\frac{1}{2}}P_{34}}{[21];\,1\,[1^2]\,3;\,[1^3]} \right\rangle. \tag{7·28}$$

Young operators for S_5

$p = [4]$

	$\langle[4];[4]\rangle$	$\left\langle\genfrac{}{}{0pt}{}{2P_{45}}{[4];[3]1;[4]}\right\rangle$	
$(\frac{5}{1})^{\frac{1}{2}}\, o^{[5]}_{[4]1;[4]1}$	$(\frac{1}{5})^{\frac{1}{2}}$	$(\frac{4}{5})^{\frac{1}{2}}$	$(\frac{5}{1})^{\frac{1}{2}}\, o^{[1^5]}_{[1^4]5;[1^4]5}$
$(\frac{5}{4})^{\frac{1}{2}}\, o^{[41]}_{[4]2;[4]2}$	$(\frac{4}{5})^{\frac{1}{2}}$	$-(\frac{1}{5})^{\frac{1}{2}}$	$(\frac{5}{4})^{\frac{1}{2}}\, o^{[21^3]}_{[1^4]1;[1^4]1}$
	$\langle[1^4];[1^4]\rangle$	$\left\langle\genfrac{}{}{0pt}{}{-2P_{45}}{[1^4];[1^3]4;[1^4]}\right\rangle$	$p = [1^4]$

(7·29)

$p = [31]$

	$\langle[31];[31]\rangle$	$\left\langle\genfrac{}{}{0pt}{}{(\frac{8}{3})^{\frac{1}{2}}P_{45}}{[31];[21]1;[31]}\right\rangle$	$\left\langle\genfrac{}{}{0pt}{}{(\frac{4}{3})^{\frac{1}{2}}P_{45}}{[31];[3]2;[31]}\right\rangle$	
$(\frac{15}{4})^{\frac{1}{2}}\, o^{[41]}_{[31]1;[31]1}$	$(\frac{12}{45})^{\frac{1}{2}}$	$(\frac{32}{45})^{\frac{1}{2}}$	$(\frac{1}{45})^{\frac{1}{2}}$	$(\frac{15}{4})^{\frac{1}{2}}\, o^{[21^3]}_{[21^2]4;[21^2]}$
$(\frac{15}{5})^{\frac{1}{2}}\, o^{[32]}_{[31]2;[31]2}$	$(\frac{15}{45})^{\frac{1}{2}}$	$-(\frac{10}{45})^{\frac{1}{2}}$	$(\frac{20}{45})^{\frac{1}{2}}$	$(\frac{15}{5})^{\frac{1}{2}}\, o^{[2^21]}_{[21^2]2;[21^2]}$
$(\frac{15}{6})^{\frac{1}{2}}\, o^{[31^2]}_{[31]3;[31]3}$	$(\frac{18}{45})^{\frac{1}{2}}$	$-(\frac{3}{45})^{\frac{1}{2}}$	$-(\frac{24}{45})^{\frac{1}{2}}$	$(\frac{15}{6})^{\frac{1}{2}}\, o^{[31^2]}_{[21^2]1;[21^2]}$
	$\langle[21^2];[21^2]\rangle$	$\left\langle\genfrac{}{}{0pt}{}{-(\frac{8}{3})^{\frac{1}{2}}P_{45}}{[21^2];[21]3;[21^2]}\right\rangle$	$\left\langle\genfrac{}{}{0pt}{}{-(\frac{4}{3})^{\frac{1}{2}}P_{45}}{[21^2];[1^3]1;[21^2]}\right\rangle$	$p = [21^2$

(7·30

$p = [2^2]$

	$\langle[2^2];[2^2]\rangle$	$\left\langle\genfrac{}{}{0pt}{}{2P_{45}}{[2^2];[21]2;[2^2]}\right\rangle$
$(\frac{10}{5})^{\frac{1}{2}}\, o^{[32]}_{[2^2]1;[2^2]1}$	$(\frac{1}{2})^{\frac{1}{2}}$	$(\frac{1}{2})^{\frac{1}{2}}$
$(\frac{10}{5})^{\frac{1}{2}}\, o^{[2^21]}_{[2^2]3;[2^2]3}$	$(\frac{1}{2})^{\frac{1}{2}}$	$-(\frac{1}{2})^{\frac{1}{2}}$

(7·31)

$p' = [3]$

$$(\tfrac{15}{4})^{\frac{1}{2}}\, o^{[41]}_{[4]2;[31]1} = \left\langle\genfrac{}{}{0pt}{}{2P_{45}}{[4];\,1\,[3]\,2;\,[31]}\right\rangle;$$

$$(\tfrac{5}{4})^{\frac{1}{2}}\, o^{[41]}_{[31]1;[4]2} = \left\langle\genfrac{}{}{0pt}{}{(\frac{4}{3})^{\frac{1}{2}}P_{45}}{[31];\,2\,[3]\,1;\,[4]}\right\rangle;$$

$p' = [1^3]$

$$(\tfrac{15}{4})^{\frac{1}{2}}\, o^{[21^3]}_{[1^4]1;[21^2]4} = \left\langle\genfrac{}{}{0pt}{}{2P_{45}}{[1^4];\,4\,[1^3]\,1;\,[21^2]}\right\rangle;$$

$$(\tfrac{5}{4})^{\frac{1}{2}}\, o^{[21^3]}_{[21^2]4;[1^4]1} = \left\langle\genfrac{}{}{0pt}{}{(\frac{4}{3})^{\frac{1}{2}}P_{45}}{[21^2];\,1\,[1^3]\,4;\,[1^4]}\right\rangle$$

(7·32)

$p' = [21]$

$$(\tfrac{10}{5})^{\frac{1}{2}}\, o^{[32]}_{[31]2;[2^2]1} = \left\langle\genfrac{}{}{0pt}{}{(\frac{8}{3})^{\frac{1}{2}}P_{45}}{[31];\,1\,[21]\,2;\,[2^2]}\right\rangle; \qquad (\tfrac{10}{5})^{\frac{1}{2}}\, o^{[2^21]}_{[21^2]2;[2^2]3} = \left\langle\genfrac{}{}{0pt}{}{(\frac{8}{3})^{\frac{1}{2}}P_{45}}{[21^2];\,3\,[21]\,2;\,[2^2]}\right\rangle;$$

$$(\tfrac{15}{5})^{\frac{1}{2}}\, o^{[32]}_{[2^2]1;[31]2} = \left\langle\genfrac{}{}{0pt}{}{2P_{45}}{[2^2];\,2\,[21]\,1;\,[31]}\right\rangle; \qquad (\tfrac{15}{5})^{\frac{1}{2}}\, o^{[2^21]}_{[2^2]3;[21^2]2} = \left\langle\genfrac{}{}{0pt}{}{2P_{45}}{[2^2];\,2[21]\,3;\,[21^2]}\right\rangle;$$

$$(\tfrac{15}{6})^{\frac{1}{2}}\, o^{[31^2]}_{[31]3;[2^2]1} = \left\langle\genfrac{}{}{0pt}{}{(\frac{8}{3})^{\frac{1}{2}}P_{45}}{[31];\,1\,[21]\,3;\,[21^2]}\right\rangle; \qquad (\tfrac{15}{6})^{\frac{1}{2}}\, o^{[31^2]}_{[21^2]1;[31]3} = \left\langle\genfrac{}{}{0pt}{}{(\frac{8}{3})^{\frac{1}{2}}P_{45}}{[21^2];\,3\,[21]\,1;\,[31]}\right\rangle.$$

(7·33)

Young operators for S_6

$p = [5]$

	$\langle[5];[5]\rangle$	$\left\langle\genfrac{}{}{0pt}{}{5^{\frac{1}{2}}P_{56}}{[5];[4]1;[5]}\right\rangle$	
$(\frac{6}{1})^{\frac{1}{2}}\, o^{[6]}_{[5]1;[5]1}$	$(\frac{1}{6})^{\frac{1}{2}}$	$(\frac{5}{6})^{\frac{1}{2}}$	$(\frac{6}{1})^{\frac{1}{2}}\, o^{[1^6]}_{[1^5]6;[1^5]6}$
$(\frac{6}{5})^{\frac{1}{2}}\, o^{[51]}_{[5]2;[5]2}$	$(\frac{5}{6})^{\frac{1}{2}}$	$-(\frac{1}{6})^{\frac{1}{2}}$	$(\frac{6}{5})^{\frac{1}{2}}\, o^{[21^4]}_{[1^5]1;[1^5]1}$
	$\langle[1^5];[1^5]\rangle$	$\left\langle\genfrac{}{}{0pt}{}{-5^{\frac{1}{2}}P_{56}}{[1^5];[1^4]5;[1^5]}\right\rangle$	$p = [1^5]$.

(7·34)

$p = [41]$

	$\langle [41]; [41] \rangle$	$\left\langle {(\frac{15}{4})^{\frac12} P_{56} \atop [41]; [31]\,1; [41]} \right\rangle$	$\left\langle {(\frac{5}{4})^{\frac12} P_{56} \atop [41]; [4]\,2; [41]} \right\rangle$	
$(\frac{24}{5})^{\frac12} o^{[51]}_{[41]1;[41]1}$	$(\frac{20}{96})^{\frac12}$	$(\frac{75}{96})^{\frac12}$	$(\frac{1}{96})^{\frac12}$	$(\frac{24}{5})^{\frac12} o^{[21^4]}_{[21^3]5;[21^3]5}$
$(\frac{24}{9})^{\frac12} o^{[42]}_{[41]2;[41]2}$	$(\frac{36}{96})^{\frac12}$	$-(\frac{15}{96})^{\frac12}$	$(\frac{45}{96})^{\frac12}$	$(\frac{24}{9})^{\frac12} o^{[2^21^2]}_{[21^3]2;[21^3]2}$
$(\frac{24}{10})^{\frac12} o^{[41^2]}_{[41]3;[41]3}$	$(\frac{40}{96})^{\frac12}$	$-(\frac{6}{96})^{\frac12}$	$-(\frac{50}{96})^{\frac12}$	$(\frac{24}{10})^{\frac12} o^{[31^3]}_{[21^3]1;[21^3]1}$
	$\langle [21^3]; [21^3] \rangle$	$\left\langle {-(\frac{15}{4})^{\frac12} P_{56} \atop [21^3]; [21^2]\,4; [21^3]} \right\rangle$	$\left\langle {-(\frac{5}{4})^{\frac12} P_{56} \atop [21^3]; [1^4]\,1; [21^3]} \right\rangle$	$p = [21^3].$

(7·35)

$p = [32]$

	$\langle [32]; [32] \rangle$	$\left\langle {2^{\frac12} P_{56} \atop [32]; [2^2]\,1; [32]} \right\rangle$	$\left\langle {3^{\frac12} P_{56} \atop [32]; [31]\,2; [32]} \right\rangle$	
$(\frac{30}{9})^{\frac12} o^{[42]}_{[32]1;[32]1}$	$(\frac{9}{30})^{\frac12}$	$(\frac{18}{30})^{\frac12}$	$(\frac{3}{30})^{\frac12}$	$(\frac{30}{9})^{\frac12} o^{[2^21^2]}_{[2^21]4;[2^21]4}$
$(\frac{30}{5})^{\frac12} o^{[3^2]}_{[32]2;[32]2}$	$(\frac{5}{30})^{\frac12}$	$-(\frac{10}{30})^{\frac12}$	$(\frac{15}{30})^{\frac12}$	$(\frac{30}{5})^{\frac12} o^{[2^3]}_{[2^21]3;[2^21]3}$
$(\frac{30}{16})^{\frac12} o^{[321]}_{[32]3;[32]3}$	$(\frac{16}{30})^{\frac12}$	$-(\frac{2}{30})^{\frac12}$	$-(\frac{12}{30})^{\frac12}$	$(\frac{30}{16})^{\frac12} o^{[321]}_{[2^21]1;[2^21]1}$
	$\langle [2^21]; [2^21] \rangle$	$\left\langle {-2^{\frac12} P_{56} \atop [2^21]; [2^2]\,3; [2^21]} \right\rangle$	$\left\langle {-3^{\frac12} P_{56} \atop [2^21]; [21^2]\,2; [2^21]} \right\rangle$	$p = [2^21].$

(7·36)

$p = [31^2]$

	$\langle [31^2]; [31^2] \rangle$	$\left\langle {(\frac{5}{2})^{\frac12} P_{56} \atop [31^2]; [21^2]\,1; [31^2]} \right\rangle$	$\left\langle {(\frac{5}{2})^{\frac12} P_{56} \atop [31^2]; [31]\,3; [31^2]} \right\rangle$
$(\frac{36}{10})^{\frac12} o^{[41^2]}_{[31^2]1;[31^2]1}$	$(\frac{10}{36})^{\frac12}$	$(\frac{25}{36})^{\frac12}$	$(\frac{1}{36})^{\frac12}$
$(\frac{36}{16})^{\frac12} o^{[321]}_{[31^2]2;[31^2]2}$	$(\frac{16}{36})^{\frac12}$	$-(\frac{10}{36})^{\frac12}$	$(\frac{10}{36})^{\frac12}$
$(\frac{36}{10})^{\frac12} o^{[31^3]}_{[31^2]4;[31^2]4}$	$(\frac{10}{36})^{\frac12}$	$-(\frac{1}{36})^{\frac12}$	$-(\frac{25}{36})^{\frac12}$

(7·37)

$p' = [4]$

$$(\tfrac{24}{5})^{\frac12} o^{[51]}_{[5]2;[41]1} = \left\langle {5^{\frac12} P_{56} \atop [5];\ 1\,[4]\,2;\ [41]} \right\rangle;$$

$$(\tfrac{6}{5})^{\frac12} o^{[51]}_{[41]1;[5]2} = \left\langle {(\frac{5}{4})^{\frac12} P_{56} \atop [41];\ 2\,[4]\,1;\ [5]} \right\rangle;$$

$p' = [1^4]$

$$(\tfrac{24}{5})^{\frac12} o^{[21^4]}_{[1^5]1;[21^3]5} = \left\langle {5^{\frac12} P_{56} \atop [1^5];\ 5\,[1^4]\,1;\ [21^3]} \right\rangle;$$

$$(\tfrac{6}{5})^{\frac12} o^{[21^4]}_{[21^3]5;[1^5]1} = \left\langle {(\frac{5}{4})^{\frac12} P_{56} \atop [21^3];\ 1\,[1^4]\,5;\ [1^5]} \right\rangle. \qquad (7·38)$$

$p' = [31]$

$$(\tfrac{30}{9})^{\frac12} o^{[42]}_{[41]2;[32]1} = \left\langle {(\frac{15}{4})^{\frac12} P_{56} \atop [41];\ 1\,[31]\,2;\ [32]} \right\rangle;$$

$$(\tfrac{24}{9})^{\frac12} o^{[42]}_{[32]1;[41]2} = \left\langle {3^{\frac12} P_{56} \atop [32];\ 2\,[31]\,1;\ [41]} \right\rangle;$$

$$(\tfrac{36}{10})^{\frac12} o^{[41^2]}_{[41]3;[31^2]1} = \left\langle {(\frac{15}{4})^{\frac12} P_{56} \atop [41];\ 1\,[31]\,3;\ [31^2]} \right\rangle;$$

$$(\tfrac{24}{10})^{\frac12} o^{[41^2]}_{[31^2]1;[41]3} = \left\langle {(\frac{5}{2})^{\frac12} P_{56} \atop [31]^2;\ 3\,[31]\,1;\ [41]} \right\rangle;$$

$$(\tfrac{36}{10})^{\frac12} o^{[321]}_{[32]3;[31^2]2} = \left\langle {3^{\frac12} P_{56} \atop [32];\ 2\,[31]\,3;\ [31^2]} \right\rangle;$$

$$(\tfrac{30}{16})^{\frac12} o^{[321]}_{[31^2]2;[32]3} = \left\langle {(\frac{5}{2})^{\frac12} P_{56} \atop [31^2];\ 3\,[31]\,2;\ [32]} \right\rangle;$$

$p' = [21^2]$

$$(\tfrac{30}{9})^{\frac12} o^{[2^21^2]}_{[21^3]2;[2^21]4} = \left\langle {(\frac{15}{4})^{\frac12} P_{56} \atop [21^3];\ 4\,[21^2]\,2;\ [2^21]} \right\rangle;$$

$$(\tfrac{24}{9})^{\frac12} o^{[2^21^2]}_{[2^21]4;[21^3]2} = \left\langle {3^{\frac12} P_{56} \atop [2^21];\ 2\,[21^2]\,4;\ [21^3]} \right\rangle;$$

$$(\tfrac{36}{10})^{\frac12} o^{[31^3]}_{[21^3]1;[31^2]4} = \left\langle {(\frac{15}{4})^{\frac12} P_{56} \atop [21^3];\ 4\,[21^2]\,1;\ [31^2]} \right\rangle;$$

$$(\tfrac{24}{10})^{\frac12} o^{[31^3]}_{[31^2]4;[21^3]1} = \left\langle {(\frac{5}{2})^{\frac12} P_{56} \atop [31^2];\ 1\,[21^2]\,4;\ [21^3]} \right\rangle;$$

$$(\tfrac{36}{10})^{\frac12} o^{[321]}_{[2^21]1;[31^2]2} = \left\langle {3^{\frac12} P_{56} \atop [2^21];\ 2\,[21^2]\,1;\ [31^2]} \right\rangle;$$

$$(\tfrac{30}{16})^{\frac12} o^{[321]}_{[31^2]2;[2^21]1} = \left\langle {(\frac{5}{2})^{\frac12} P_{56} \atop [31^2];\ 1\,[21^2]\,2;\ [2^21]} \right\rangle.$$

(7·39)

$p' = [2^2]$

$$\left(\tfrac{30}{16}\right)^{\frac{1}{2}} o^{[321]}_{[32]3;\,[2^21]1} = \left\langle \begin{matrix} 2^{\frac{1}{2}} P_{56} \\ [32];\; 1\,[2^2]\,3;\; [2^21] \end{matrix} \right\rangle; \quad \left(\tfrac{30}{16}\right)^{\frac{1}{2}} o^{[321]}_{[2^21]1;\,[32]3} = \left\langle \begin{matrix} 2^{\frac{1}{2}} P_{56} \\ [2^21];\; 3\,[22]\,1;\; [32] \end{matrix} \right\rangle. \tag{7·40}$$

From the last equation we read, for example, that

$$\left(\tfrac{30}{16}\right)^{\frac{1}{2}} o^{[321]}_{121231;\,121123} = o^{[2^21]}_{12123;\,11223}\, 2^{\frac{1}{2}} P_{56}\, o^{[32]}_{11221;\,12112}, \tag{7·41}$$

whilst from (7·36) we read

$$\left(\tfrac{30}{16}\right)^{\frac{1}{2}} o^{[321]}_{121231;\,112321} = \left(\tfrac{16}{30}\right)^{\frac{1}{2}} o^{[2^21]}_{12123;\,11232} + \left(\tfrac{2}{30}\right)^{\frac{1}{2}} o^{[2^21]}_{12123;\,11223}\, 2^{\frac{1}{2}} P_{56}\, o^{[2^21]}_{11223;\,11232} + \left(\tfrac{12}{30}\right)^{\frac{1}{2}} o^{[2^21]}_{12123;\,12132}\, 3^{\frac{1}{2}} P_{56}\, o^{[2^21]}_{12132;\,11232}. \tag{7·42}$$

Here

$$121231 \equiv \begin{pmatrix} 136 \\ 24 \\ 5 \end{pmatrix}, \quad 121123 \equiv \begin{pmatrix} 134 \\ 25 \\ 6 \end{pmatrix}, \quad 112321 \equiv \begin{pmatrix} 126 \\ 35 \\ 4 \end{pmatrix}, \tag{7·43}$$

are three of the 16 possible Yamanouchi symbols, or standard arrangements, for the representation $R^{[321]}$ of S_6 and the above are just two examples of the $16^2 = 256$ Young operators for $R^{[321]}$, all of which may be deduced from appropriate entries in the above tables (viz. since $f^{[32]} = f^{[2^21]} = 5$ we obtain 50 of these operators from (7·36), since $f^{[31^2]} = 6$ we obtain 36 of the operators from (7·37), similarly 120 from (7·39) and finally 50 from (7·40): total 256).

Conclusions

Young operator expansions of the type considered here have been considered implicitly by Hassitt (1955) in connexion with fractional parentage coefficient theory, but the explicit general expressions obtained here are new. This work has been extended to the case of the alternative representation form (useful in physical applications) in which $P_{A,\,A+1}$ is represented by a diagonal matrix; this will be treated in part II. In the appendix to the present paper explicit formulae are given for the general case.

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Appendix. General explicit formulae

Since the completion of the work described in the body of this paper, explicit expressions for the coefficients c^p_{0t}, c^p_{st} have been obtained directly in terms of Young axial distances. These new formulae may be regarded as a generalization of the explicit formulae obtained in the Wigner supermultiplet case to the case of the most general representation of the symmetric group and were in fact obtained in a similar manner through use of the general dimension formula. Alternatively, the Wigner supermultiplet case results may be obtained simply by specialization from these general formulae.

As before we let

$$p = [(p^{(1)})^{k_1}(p^{(2)})^{k_2}\dots(p^{(n)})^{k_n}] \tag{A 1}$$

denote a regular partition of A with possible repetitions

$$A = k_1p^{(1)}+k_2p^{(2)}+\dots+k_np^{(n)}, \tag{A 2}$$

with $$p^{(1)} > p^{(2)} > \dots > p^{(n)} > 0;\quad k_1 \geqslant 1,\ k_2 \geqslant 1,\ \dots,\ k_n \geqslant 1. \tag{A 3}$$

(The considerations of § 7 show, however, that some of the k_r could be zero.) To this partition corresponds a regular Young tableau of A squares. Any square of this tableau which may be removed to leave a regular tableau of $A-1$ squares is called an s-labelled square for p and denoted by $\{p\}_s$, with s taking on the n values

$$s_r = k_1+k_2+\dots+k_r \quad (r = 1, 2, \dots, n)$$
$$= \text{the tableau row number of the square } \{p\}_s. \tag{A 4}$$

Any square which may be added to the tableau of p to form a regular tableau with $A+1$ squares is called a t-labelled square for p and denoted by $\{p\}^t$, with t taking on the $n+1$ values

$$t_q = k_0+k_1+k_2+\dots+k_q \quad (q = 0, 1, 2, \dots, n;\ k_0 = 1)$$
$$= \text{the tableau row number of the square } \{p\}^t. \tag{A 5}$$

We let a^p_{ts} denote the Young axial distance (positive horizontally to the left or vertically downwards) from square $\{p\}^t$ to square $\{p\}_s$. (This is a change in notation from the body of the paper: note the reversed order of the indices t and s.) Similarly we let $d^p_{tt'}$ denote the Young axial distance from square $\{p\}^t$ to square $\{p\}^{t'}$ and $b^p_{ss'}$ the axial distance from $\{p\}_s$ to $\{p\}_{s'}$. (b and d are used here instead of a to avoid confusion when numerical values for the row numbers are inserted.) We note as before that

$$a^p_{ts} = 1 \quad \text{for} \quad t = s, \tag{A 6}$$

for $\{p\}^t$ is then adjacent to $\{p\}_s$ on the same row. It is convenient to introduce the purely formal conventions

$$d^p_{tt'} = 1 \quad \text{for} \quad t = t';\quad b^p_{ss'} = 1 \quad \text{for} \quad s = s'. \tag{A 7}$$

In terms of these axial distances, or their moduli, the new formulae (using the above conventions) are as follows

$$c^p_{0t} = [f^{pt}/\{(A+1)f^p\}]^{\frac{1}{2}} = \sqrt{\left\{\frac{\prod_s |a^p_{ts}|}{\prod_{t'} |d^p_{tt'}|}\right\}}, \tag{A 8}$$

$$c^p_{st} = \frac{c^p_{0t}}{a^p_{ts}c^{p/s}_{0s}} = \{\text{sign } a^p_{ts}\}\sqrt{\left\{\frac{\prod_{s'\neq s} |a^p_{ts'}| \prod_{t'\neq t} |a^p_{t's}|}{\prod_{t'} |d^p_{tt'}| \prod_{s'} |b^p_{ss'}|}\right\}}, \tag{A 9}$$

where
$$\{\text{sign } a^p_{ts}\} = \begin{cases} +1 & \text{if} \quad t < s, \\ -1 & \text{if} \quad t > s. \end{cases} \tag{A 10}$$

(Note that the cancellation of the factor $1/a^p_{ts}$ in the formula for c^p_{ts} is now apparent: $\sqrt{|a^p_{ts}|}$ is a factor of both c^p_{0t} and $1/c^{p/s}_{0t}$.)

We have thus proved, in the body of the paper, that the coefficients (A 8) and (A 9) set up in the form of an $(n+1)$-rowed square array

$$\begin{bmatrix} c^p_{0t} \\ c^p_{st} \end{bmatrix} \tag{A 11}$$

form an orthogonal matrix.

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